4.

X : 女性最高的排名。

$$P(X=1)=P(\underbrace{女}_{1st},其餘任意排列)=\frac{\overbrace{C_1^5}^{5女取1}\cdot 9!}{10!}=\frac{5}{10}=\frac{1}{2}$$

$$P(X=2)=P(\underbrace{男女}_{1st,2nd},其餘任意排列)=\frac{\overbrace{C_1^5}^{5男取1}\cdot\overbrace{C_1^5}^{5女取1}\cdot 8!}{10!}=\frac{5\times 5}{10\times 9}=\frac{5}{18}$$

$$P(X=3)=P(\underbrace{男男女}_{1st,2nd,3rd},其餘任意排列)=\frac{\overbrace{C_2^5\times 2!}^{5男取2且排列}\cdot\overbrace{C_1^5}^{5女取1}\cdot 7!}{10!}=\frac{5\times 4\times 5}{10\times 9\times 8}=\frac{5}{36}$$

$$P(X=4)=P(\underbrace{男男男女}_{1st,2nd,3rd,4th},其餘任意排列)=\frac{\overbrace{C_3^5\times 3!}^{5男取3且排列}\cdot\overbrace{C_1^5}^{5女取1}\cdot 6!}{10!}=\frac{5\times 4\times 3\times 5}{10\times 9\times 8\times 7}=\frac{5}{84}$$

$$P(X=5)=P(\underbrace{男男男男女}_{1st,2nd,3rd,4th,5th},其餘任意排列)=\frac{\overbrace{C_4^5\times 4!}^{5男取4且排列}\cdot\overbrace{C_1^5}^{5女取1}\cdot 5!}{10!}=\frac{5\times 4\times 3\times 2\times 5}{10\times 9\times 8\times 7\times 6}=\frac{5}{84}=\frac{5}{252}$$

$$P(X=6)=P(\underbrace{男男男男男女女女女女}_{1st,2nd,3rd,4th,5th,6th,\cdots,10th})=\frac{5!\times 5!}{10!}=\frac{5\times 4\times 3\times 2\times 1}{10\times 9\times 8\times 7\times 6}=\frac{1}{252}$$

$$P(X=7)=\cdots=P(X=10)=0$$

14.

X : 玩家 1 所贏的場數。

$$P(X=0)=P(\text{玩家 1 輸玩家 2})$$

$$=\frac{\overbrace{4!}^{玩家1的數爲1,其餘數任意排列}+\overbrace{C_1^3}^{玩家1,2的數之組合\ (2,3),(2,4),(2,5)}\times 3!+\overbrace{C_1^2}^{玩家1,2的數之組合\ (3,4),(3,5)}\times 3!+\overbrace{1}^{玩家1,2的數之組合\ (4,5)}\times 3!}{5!}=\frac{1}{2}$$

$$P(X=1)=P(\text{玩家 1 贏玩家 2，但輸玩家 3})$$

$$=P\left\{\begin{array}{l}玩家1,2,3的數之組合:(2,1,3),(2,1,4),(2,1,5),(3,1,4),(3,1,5),\\(4,1,5),(3,2,4),(3,2,5),(4,2,5),(4,3,5),其餘數任意排列在後\end{array}\right\}$$

$$=\frac{C_1^{10}\times 2!}{5!}=\frac{1}{6}$$

$$P(X=2)=P(\text{玩家 1 贏玩家 2 , 3，但輸玩家 4})$$

$$=P\left\{\begin{matrix}\text{玩家1,2,3,4的數之組合}:(3,2,1,4),(3,2,1,5),(3,1,2,4),(3,1,2,5),\\(4,1,2,5),(4,1,3,5),(4,2,1,5),(4,2,3,5),(4,3,1,5),(4,3,2,5)\end{matrix}\right\}$$

$$=\frac{C_1^{10}\times 1}{5!}=\frac{1}{12}$$

$$P(X=3)=P(\text{玩家 1 贏玩家 2 , 3 ,4 ，但輸玩家 5})$$

$$=P\left\{\begin{matrix}\text{玩家1,2,3,4,5的數之組合}:(4,1,2,3,5),(4,1,3,2,5),(4,2,1,3,5),\\(4,2,3,1,5),(4,3,1,2,5),(4,3,2,1,5)\end{matrix}\right\}$$

$$=\frac{6}{5!}=\frac{1}{20}$$

$$P(X=4)=\frac{\overbrace{4!}^{\text{玩家1的數爲5,其餘任意排列}}}{5!}=\frac{1}{5}$$

17.

(a)

$$P(X=1)=P(X\le 1)-P(X<1)=F(1)-F(1^-)=\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$$

$$P(X=2)=P(X\le 2)-P(X<2)=F(2)-F(2^-)=\frac{11}{12}-(\frac{1}{2}+\frac{1}{4})=\frac{1}{6}$$

$$P(X=3)=P(X\le 3)-P(X<3)=F(3)-F(3^-)=1-\frac{11}{12}=\frac{1}{12}$$

(b)

$$P(\frac{1}{2}<X<\frac{3}{2})=P(\frac{1}{2}<X\le 1)+P(1<X<\frac{3}{2})$$

$$=P(X\le 1)-P(X\le\frac{1}{2})+P(X<\frac{3}{2})-P(X\le 1)$$

$$=F\left((\frac{3}{2})^-\right)-F\left(\frac{1}{2}\right)=\left(\frac{1}{2}+\frac{\frac{3}{2}-1}{4}\right)-\frac{\frac{1}{2}}{4}=\frac{1}{2}$$

20.

X：在停止遊戲時，賭客所贏得的獎金。

$Case1$：第1場贏 $\Rightarrow$ 贏1元

$$Case2: \text{第1場輸} \Rightarrow \begin{Bmatrix} (L,W,W) \Rightarrow \text{贏1元} \\ (L,W,L) \Rightarrow \text{輸1元} \\ (L,L,W) \Rightarrow \text{輸1元} \\ (L,L,L) \Rightarrow \text{輸3元} \end{Bmatrix}$$

(a)

$$P(X>0)=\frac{18}{38}+\frac{20}{38}\left(\frac{18}{38}\right)^2=0.5918$$

(b)

贏少輸多，所以不能保證此爲贏的策略。

(c)

$$E(X)=1\cdot P(X=1)+(-1)\cdot P(X=-1)+(-3)\cdot P(X=-3)$$
$$=1\cdot\left[\frac{18}{38}+\frac{20}{38}\left(\frac{18}{38}\right)^2\right]+(-1)\cdot\left[2\times\left(\frac{20}{38}\right)^2\times\frac{18}{38}\right]+(-3)\cdot\left(\frac{20}{38}\right)^3=-0.108$$

22.

X：比賽的場數。

(a)

$$P(X=2)=\overbrace{p^2}^{A\,win}+\overbrace{(1-p)^2}^{B\,win}$$

$$P(X=3)=\overbrace{p\cdot(1-p)}^{AB}+\overbrace{(1-p)\cdot p}^{BA}=2p(1-p)$$

$*$前2場的組合需爲 AB or BA 才會進入第3場

$$E(X)=2\cdot P(X=2)+3\cdot P(X=3)$$
$$=2\cdot\left[p^2+(1-p)^2\right]+3\cdot\left[2p(1-p)\right]=-2p^2+2p+2$$

$$\frac{dE(X)}{dp}=-4p+2\overset{Let}{=}0\ \Rightarrow\ p=\frac{1}{2}$$

(b)

$$P(X=3)=\overbrace{p^3}^{A\,win}+\overbrace{(1-p)^3}^{B\,win}$$

$$P(X=4)=\overbrace{\frac{3!}{2!}}^{\text{前3場的排列數}}\cdot p^2(1-p)\cdot p+\overbrace{\frac{3!}{2!}}^{\text{前3場的排列數}}\cdot(1-p)^2\cdot p\cdot(1-p)$$
$$=-6p^4+12p^3-9p^2+3p$$

$*$前3場的組合需爲 $2A1B$ 且第4場爲 A 贏 or 前3場組合爲 $2B1A$ 且第4場爲 B 贏

$$P(X=5)=\overbrace{\frac{4!}{2!2!}}^{\text{前4場的排列數}}\cdot p^2(1-p)^2$$

$*$前4場的組合需爲$2A2B$,才會進入第5場

$$E(X)=3\cdot P(X=3)+4\cdot P(X=4)+5\cdot P(X=5)$$
$$=3\cdot\left[p^3+(1-p)^3\right]+4\cdot\left[-6p^4+12p^3-9p^2+3p\right]+5\cdot\left[6p^2(1-p)^2\right]$$
$$=3\cdot\left(2p^4-4p^3+p^2+p+1\right)$$

$$\frac{dE(X)}{dp}=3\cdot\left(8p^3-12p^2+2p+1\right)\overset{Let}{=}0\Rightarrow p=\frac{1}{2}\ or$$
$$p=\frac{1\pm\sqrt{2}}{2}\ (\text{不合})$$

38.

$$E(X^2)=Var(X)+\left[E(X)\right]^2=5+1=6$$

(a)

$$E\left[(2+X)^2\right]=E\left(4+4X+X^2\right)=4+4E(X)+E(X^2)=4+4+6=14$$

(b)

$$Var(4+3X)=9Var(X)=45$$

Theoretical Exercises

2.

$\because F_X(x)=P(X\le x)$

$Let\ Y=e^x$

$$F_Y(y)=P(Y\le y)=P(e^X\le y)=P(X\le\ln y)=F_X(\ln y)$$

4.

$$E(N)=\sum_{n=0}^{\infty}nP(N=n)=\sum_{n=1}^{\infty}nP(N=n)=\sum_{n=1}^{\infty}\sum_{k=1}^{n}1\cdot P(N=n)\qquad(\because\sum_{k=1}^{n}1=n)$$
$$=\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}P(N=n)=\sum_{k=1}^{\infty}P(N\ge k)=\sum_{i=1}^{\infty}P(N\ge i)$$