NTHU MATH 2810 Final Examination Jan 8, 2008

Note. There are 8 problems in total. The total score is 100pts. **To ensure consideration for partial scores, write down intermediate steps where necessary.**

1. (*16pts, 2pts for each*) For the following statements, please answer true or false. If false, please explain why.

   (a) Let X be a continuous random variable, then $P(X \in A) = 0$ for any countable set A.

   (b) For a continuous random variable, the values of its probability density function (pdf) must be between 0 and 1.

   (c) Transformation by using Jacobian can be applied to find the joint pdf when the mapping between two groups of n random variables is not one-to-one.

   (d) A random variable X with possible values 0 and 1 will have $E(X^k) = E(X)$ for $k = 2, 3, 4, \ldots$.

   (e) Let $X_1, \ldots, X_n$ be i.i.d. from a distribution with finite variance. The variance of $\overline{X}_n = (X_1 + \cdots + X_n)/n$ always tends to zero as the sample size n increases to infinity.

   (f) The correlation coefficient of two independent random variables is zero.

   (g) If X and Y are uncorrelated, then $E(X|Y) = E(X)$.

   (h) If X and Y are independent, then $E(XY) = E(X)E(Y)$ and $E(X/Y) = E(X)/E(Y)$.

2. (*15pts, 3pts for each*) For each of the random variables X below, determine the type of distribution (i.e., Normal, Exponential, Gamma, Beta, Uniform, Poisson, Hypergeometric, Binomial, Bernoulli, Negative binomial, Geometric, etc.) which best models X and give the values of the parameters of the distribution chosen.

   (a) The average height of professors at a certain college is 68 inches, and the mean squared deviation from this average (i.e., variance) is 2. Let X be the height of a randomly chosen professor.

   (b) As part of a grand opening promotion, a department store has advertised that every 1000^{th} purchase made on opening day will be given to the customer for free. The store expects 5 purchases to be made every minute. Let X be the time (in minites) from opening until the first free purchase is given away.

   (c) A doctor sees an average of 2 patients with a certain non-contagious disease per year. Let X be the number of patients with this disease he sees in the next 2 years.

   (d) A fraction 1/8 of all people are left-handed. There are 20 people at a party. Let X be the number of left-handed people.

   (e) A pencil is dropped on a table in a random direction. Let X be the angle (in degrees, direction matters) between the direction which the pencil points and north.

3. (*6pts*) Five hundred independent rolls of a fair die will be made. What is the approximate probability that the outcome "5" will occur at least 100 times? Use Φ to express your answer, where Φ is the cumulative distribution function (cdf) of Normal(0, 1) distribution.

   [**Hint.** Use the Normal approximation to Binomial distribution. The mean and variance of Binomial(n, p) are np and $np(1-p)$, respectively.]

4. Consider a line segment of length L, which we will consider to be the inverval $[0, L]$ in the real line. Let X and Y be independent random points on this line segment, where X is uniformly distributed on $[0, L/2]$ and Y is uniformly distributed on the interval $[L/2, L]$.

   (a) (*2pts*) Find the joint pdf of X and Y.

   (b) (*4pts*) Find the probability that X is closer to Y than to the origin.

5. Suppose that 10 married couples are randomly seated at a round table with 20 seats. Let the random variable N be the number of of wives sitting next to their husbands.

   (a) (*4pts*) Let I_i, $i = 1, \ldots, 10$, be the indicator functions where $I_i = 1$ if couple i sits together, and 0 if they do not. Show that $P(I_i = 1) = 2/19$ and $P(I_i = 1, I_j = 1) = 2/(19 \times 9)$, for $i \neq j$.

   (b) (*3pts*) What is the expectation of N?

   (c) (*5pts*) What is the variance of N?

   [**Hint.** $N = \sum_{i=1}^{10} I_i$.]

6. The Weibull(α, β) cdf is:

$$F(x) = 1 - e^{\left(\frac{x}{\alpha}\right)^{\beta}}, \quad x \geq 0, \quad \alpha > 0, \quad \beta > 0.$$

   (a) (*2pts*) Find the pdf of the Weibull distribution.

   (b) (*4pts*) What transformations can be used to generate independent Weibull random variables $X_1, \ldots, X_n$ from independent uniform random variables $U_1, \ldots, U_n$, where $U_i \sim$ Uniform$(0, 1)$, $i = 1, \ldots, n$?

   (c) (*4pts*) Find the cdf of the minimum of n independent Weibull random variables $X_1, \ldots, X_n$.

7. Let X_1 and X_2 be i.i.d. from Normal$(0, 1)$ distribution, whose pdf is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty.$$

   Define

$$W_1 = \sqrt{3}X_1 + X_2 \text{ and } W_2 = X_1 - \sqrt{3}X_2.$$

   (a) (*2pts*) Write down the joint pdf of (X_1, X_2).

   (b) (*6pts*) Compute the joint pdf of (W_1, W_2). [**Note**. $X_1^2 + X_2^2 = (W_1^2 + W_2^2)/4$.]

   (c) (*2pts*) Examine whether or not W_1 and W_2 are independent from their joint pdf.

   (d) (*6pts*) Let $Y = X_1^2$. Find the pdf of Y.

   (e) (*2pts*) The random variable Y has the same distribution as one of the Gamma(α, λ) distributions, whose pdf is

$$\frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x > 0.$$

   For which parameters (α, λ) is the Gamma distribution the same as that of Y? Explain your answer.

8. Let X and Y be the minimum and maximum of two independent Uniform$(0,1)$ random variables.

   (a) (*2pts*) Verify that the joint pdf of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} 2, & \text{if } 0 < x < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

   (b) (*2pts*) Find the marginal pdf of X and marginal pdf of Y.

   (c) (*2pts*) Find the conditional pdf of Y given $X = x$, $0 < x < 1$.

   (d) (*2pts*) Find $E(Y|X = x)$.

   (e) (*3pts*) Find $E(XY)$ by applying the law of total expectation, i.e., $E(XY) = E[E(XY|X)]$. (**Note**. Any other methods to find the solution are *not* acceptable.)

   (f) (*3pts*) Find $Var(Y|X = x)$.

   (g) (*3pts*) Compute $Var(Y)$ and $E[Var(Y|X)]$ to verify that $E[Var(Y|X)] < Var(Y)$.