1. (*16pts, 2pts for each*)
   (a) True.
   (b) False. The values of a pdf can be larger than 1. However, the integration of a pdf over any region must have values between 0 and 1.
   (c) False. It must be a one-to-one transformation.
   (d) True.
   (e) True.
   (f) True.
   (g) False. If X and Y are *independent*, then $E(X|Y) = E(X)$. Zero correlation (i.e., uncorrelated) is a weaker condition than independence. It cannot guarantee this property.
   (h) False. When X and Y are independent, $E(X/Y) = E(X)E(1/Y) \neq E(X)/E(Y)$ in general.

2. (*15pts, 3pts for each*)
   (a) Normal(μ, σ^2) with $\mu = 68$ and $\sigma^2 = 2$.
   (b) Gamma(α, λ) with $\alpha = 1000$ and $\lambda = 5$. (An alternative answer that is acceptable is Exponential(λ) with $\lambda = \frac{1}{1000/5} = \frac{1}{200}$.)
   (c) Poisson(λ) with $\lambda = 2 \times 2 = 4$.
   (d) Binomial(n, p) with $n = 20$ and $p = 1/8$.
   (e) Uniform(a, b) with $a = 0$ and $b = 360$.

3. (*6pts*) Let X be the number of "5" that occurs in the 500 rolls, then

$$X \sim \text{Binomial}(500, 1/6)$$

since the die is fair. Therefore,

$$E(X) = 500 \times (1/6) = 500/6, \quad \text{and} \quad Var(X) = 500 \times (5/36) = 2500/36.$$

Because $n = 500$ is large, we can use Normal approximation to evaluate $P(X \geq 100)$ as follows:

$$\begin{aligned} P(X \geq 100) = P(X \geq 99.5) &= P\left(\frac{X - \frac{500}{6}}{\sqrt{\frac{2500}{36}}} \geq \frac{99.5 - \frac{500}{6}}{\sqrt{\frac{2500}{36}}}\right) \\ &\approx P\left(Z \geq \frac{99.5 - \frac{500}{6}}{\sqrt{\frac{2500}{36}}}\right) = 1 - \Phi\left(\frac{99.5 - \frac{500}{6}}{\sqrt{\frac{2500}{36}}}\right) \\ &= 1 - \Phi(1.94), \end{aligned}$$

where $Z \sim$ Normal(0, 1).

4. (a) (*2pts*) Because $X \sim \text{Uniform}(0, L/2)$, $Y \sim \text{Uniform}(L/2, L)$, and they are independent, their joint pdf is

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \begin{cases} \frac{1}{L/2}\frac{1}{L/2} = \frac{4}{L^2}, & \text{for } 0 < x < L/2 \text{ and } L/2 < y < L, \\ 0, & \text{otherwise.} \end{cases}$$

(b) (*4pts*) Our condition is that

$$Y - X < X, \quad (\text{i.e., } Y < 2X).$$

The probability we want to know is thus the probability that $(X, Y) \in S$, where $S = \{(x,y) | L/2 \le y < 2x \le L\}$. Therefore,

$$P(Y < 2X) = \int\int_S f_{X,Y}(x,y)\, dxdy$$
$$= \int_{L/4}^{L/2}\int_{L/2}^{2x} \frac{4}{L^2}\, dydx = \frac{4}{L^2}\int_{L/4}^{L/2}(2x - L/2)\, dx = \frac{4}{L^2} \times \frac{L^2}{16} = \frac{1}{4}.$$

5. (a) (*4pts*) To compute $P(I_i = 1)$, assume that husband $\#i$ is seated first. Then, of the remaining 19 seats which are available at random to wife $\#i$, only two will lead to sitting together. So,

$$P(I_i = 1) = \frac{20 \times 2}{20 \times 19} = \frac{2}{19}.$$

Because

$$P(I_i = 1, I_j = 1) = P(I_j = 1 | I_i = 1)P(I_i = 1),$$

it is enough to compute $P(I_j = 1|I_i = 1)$. This is the same as having a *line* (Note. not a circle) of 18 chairs in a row, for the jth couple to choose from randomly. There are $(1 + 16 \times 2 + 1) = 34$ ways to seat the jth husband and wife next to each other out of 18×17 possible ways where they could be seated. Thus,

$$P(I_j = 1|I_i = 1) = \frac{34}{18 \times 17} = \frac{1}{9}, \quad \text{and} \quad P(I_i = 1, I_j = 1) = \frac{1}{9} \times \frac{2}{19}.$$

(b) (*3pts*) Because $N = \sum_{i=1}^{10} I_i$, by the fundamental formula about expectation,

$$E(N) = \sum_{i=1}^{10} E(I_i).$$

Now, since I_i's are indicator functions,

$$E(I_i) = P(I_i = 1) = \frac{2}{19}, \quad \text{and} \quad E(N) = 10 \times \frac{2}{19} = \frac{20}{19}.$$

(c) (*5pts*) To compute the variance of N, we use the formula:

$$Var(N) = Var\left(\sum_{i=1}^{10} I_i\right) = \sum_{i=1}^{10} Var(I_i) + 2\sum_{1 \le i < j \le 10} Cov(I_i, I_j).$$

Because

$$Var(I_i) = E(I_i^2) - [E(I_i)]^2 = \frac{2}{19} - \left(\frac{2}{19}\right)^2 = \frac{34}{361},$$

and

$$Cov(I_i, I_j) = E(I_iI_j) - E(I_i)E(I_j) = \frac{2}{9 \times 19} - \left(\frac{2}{19}\right)^2 = \frac{2}{3249},$$

we get

$$Var(N) = 10 \times \frac{34}{361} + 2 \times 45 \times \frac{2}{3249} = \frac{3240}{3249} = \frac{360}{361}.$$

6. (a) (*2pts*) The pdf of the Weibull(α, β) distribution is

$$f(x) = \frac{d}{dx}F(x) = \frac{\beta}{\alpha^\beta} x^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^\beta}$$

for $x \geq 0$, and $f(x) = 0$, for $x < 0$.

(b) (*4pts*) Notice that if $X_1, \ldots, X_n$ are i.i.d. from a continuous distribution with cdf F, then $F(X_1), \ldots, F(X_n)$ are i.i.d. $\sim$ Uniform$(0,1)$. For the case of Weibull(α, β), let $U_i = F(X_i) = 1 - e^{-\left(\frac{X_i}{\alpha}\right)^\beta}$, for $i = 1, \ldots, n$, then

$$X_i = F^{-1}(U_i) = \alpha \left[-\log(1 - U_i)\right]^{\frac{1}{\beta}}, \quad i = 1, \ldots, n,$$

are i.i.d. $\sim$ Weibull(α, β) distribution.

(c) (*4pts*) Let $X_{(1)} = \min\{X_1, \ldots, X_n\}$. For $x > 0$,

$$P(X_{(1)} > x) = P(X_1 > x, \cdots, X_n > x) = [P(X_1 > x)]^n = \left[e^{-\left(\frac{x}{\alpha}\right)^\beta}\right]^n = e^{-n\left(\frac{x}{\alpha}\right)^\beta}.$$

Therefore, the cdf of $X_{(1)}$ is

$$F_{X_{(1)}}(x) = \begin{cases} 1 - e^{-n\left(\frac{x}{\alpha}\right)^\beta}, & \text{for } x \geq 0, \\ 0, & \text{for } x < 0, \end{cases}$$

which shows that $X_{(1)} \sim$ Weibull$(\alpha n^{-\frac{1}{\beta}}, \beta)$.

7. (a) (*2pts*) Because X_1 and X_2 are independent, their joint pdf is

$$f_{X_1,X_2}(x_1, x_2) = f(x_1)f(x_2) = \frac{1}{2\pi} e^{-(x_1^2 + x_2^2)/2},$$

where $-\infty < x_1,\ x_2 < \infty$.

(b) (*6pts*) The inverse function of the transformation is:

$$X_1 = g_1^{-1}(W_1, W_2) = \frac{\sqrt{3}}{4}W_1 + \frac{1}{4}W_2 \quad \text{and} \quad X_2 = g_2^{-1}(W_1, W_2) = \frac{1}{4}W_1 - \frac{\sqrt{3}}{4}W_2.$$

Because

$$\frac{\partial g_1^{-1}}{\partial W_1} = \frac{\sqrt{3}}{4}, \quad \frac{\partial g_1^{-1}}{\partial W_2} = \frac{1}{4}, \quad \frac{\partial g_2^{-1}}{\partial W_1} = \frac{1}{4}, \quad \frac{\partial g_2^{-1}}{\partial W_1} = \frac{-\sqrt{3}}{4},$$

$$J = \begin{vmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{\sqrt{3}}{4} \end{vmatrix} = -\frac{1}{4},$$

and $X_1^2 + X_2^2 = \frac{1}{4}(W_1^2 + W_2^2)$, the joint pdf of (W_1, W_2) is:

$$\begin{aligned} f_{W_1,W_2}(w_1, w_2) &= f_{X_1,X_2}(g_1^{-1}(w_1, w_2), g_2^{-1}(w_1, w_2)) \times |J| \\ &= \frac{1}{2\pi} e^{-\frac{1}{8}(w_1^2 + w_2^2)} \times \left|-\frac{1}{4}\right| = \frac{1}{8\pi} e^{-\frac{1}{8}(w_1^2 + w_2^2)} \\ &= \left(\frac{1}{2\sqrt{2\pi}} e^{-\frac{w_1^2}{2\times 4}}\right) \times \left(\frac{1}{2\sqrt{2\pi}} e^{-\frac{w_2^2}{2\times 4}}\right), \end{aligned}$$

where $-\infty < w_1,\ w_2 < \infty$. (Note that the joint pdf is a product of two Normal pdfs.)

(c) (*2pts*) Because the joint pdf of (W_1, W_2) is proportional to a product of two terms, one depending only on w_1 and the other depending only on w_2, W_1 and W_2 are independent.

(d) (*6pts*) We can get the cdf of Y, $F_Y(y)$, for $y \geq 0$ by

$$F_Y(y) = P(Y \leq y) = P(X_1^2 \leq y) = P(-\sqrt{y} \leq X_1 \leq \sqrt{y}) = \Phi(\sqrt{y}) - \Phi(-\sqrt{y}),$$

where Φ is the cdf of Normal(0, 1). Then, the pdf of Y is

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}F_Y(y) = \frac{d}{dy}\Phi(\sqrt{y}) - \frac{d}{dy}\Phi(-\sqrt{y}) \\ &= \frac{1}{\sqrt{2\pi}}e^{-y/2}\left(\frac{1}{2\sqrt{y}}\right) - \frac{1}{\sqrt{2\pi}}e^{-y/2}\left(-\frac{1}{2\sqrt{y}}\right) = \frac{1}{\sqrt{2\pi y}}e^{-y/2}, \end{aligned}$$

for $y \geq 0$ and $f_Y(y) = 0$ for $y < 0$.

(e) (*2pts*) We can write the pdf of Y as

$$\frac{(1/2)^{1/2}}{\sqrt{\pi}} \times y^{(1/2)-1} \times e^{-(1/2)y},$$

which is the pdf of Gamma(1/2, 1/2) because $\Gamma(1/2) = \sqrt{\pi}$.

8. (a) (*2pts*) Let U_1 and U_2 be i.i.d. $\sim$ Uniform$(0, 1)$, then $X = \min(U_1, U_2)$ and $Y = \max(U_1, U_2)$. Therefore, for $0 < x < y < 1$, the joint pdf of X and Y is

$$f_{X,Y}(x, y) = (2!)f_{U_1}(x)f_{U_2}(y) = 2.$$

(b) (*2pts*) The marginal pdf of X is

$$f_X(x) = \int_x^1 f_{X,Y}(x, y)\, dy = \int_x^1 2\, dy = 2(1 - x),$$

for $0 < x < 1$, and $f_X(x) = 0$, otherwise. Similarly, the marginal pdf of Y is

$$f_Y(y) = \int_0^y f_{X,Y}(x, y)\, dx = \int_0^y 2\, dx = 2y,$$

for $0 < y < 1$, and $f_Y(y) = 0$, otherwise.

(c) (*2pts*) For a fixed $x \in (0, 1)$, the conditional pdf of Y is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \frac{1}{1 - x},$$

for $x < y < 1$ and $f_{Y|X}(y|x) = 0$, otherwise.

(d) (*2pts*)

$$E[Y|X = x] = \int_x^1 y\, f_{Y|X}(y|x)\, dy = \int_x^1 \frac{y}{1 - x}\, dy = \frac{1}{1 - x}\left(\frac{1}{2}y^2\Big|_x^1\right) = \frac{1 + x}{2},$$

for $0 < x < 1$. Therefore, $E[Y|X] = \frac{1+X}{2}$.

(e) (*3pts*)

$$\begin{aligned} E(XY) &= E_X[E_{Y|X}(XY|X)] = E_X\left\{X[E_{Y|X}(Y|X)]\right\} = E_X\left(X \times \frac{1+X}{2}\right) \\ &= \int_{-\infty}^{\infty} x \times \frac{1+x}{2} \times f_X(x)\,dx = \int_0^1 x \times \frac{1+x}{2} \times 2(1-x)\,dx \\ &= \int_0^1 (x - x^3)\,dx = \frac{1}{4}. \end{aligned}$$

(f) (*3pts*) Because

$$Var(Y|X=x) = E(Y^2|X=x) - [E(Y|X=x)]^2,$$

and

$$E[Y^2|X=x] = \int_x^1 y^2\, f_{Y|X}(y|x)\,dy = \int_x^1 \frac{y^2}{1-x}\,dy = \frac{1}{1-x}\left(\frac{1}{3}y^3\Big|_x^1\right) = \frac{1+x+x^2}{3},$$

we get

$$Var(Y|X=x) = \frac{1+x+x^2}{3} - \left(\frac{1+x}{2}\right)^2 = \frac{x^2-2x+1}{12} = \frac{(x-1)^2}{12},$$

for $0 < x < 1$.

(g) (*3pts*) The variance of Y is

$$\begin{aligned} Var(Y) &= E(Y^2) - [E(Y)]^2 = \int_{-\infty}^{\infty} y^2\, f_Y(y)\,dy - \left[\int_{-\infty}^{\infty} y\, f_Y(y)\,dy\right]^2 \\ &= \int_0^1 y^2(2y)\,dy - \left[\int_0^1 y(2y)\,dy\right]^2 = \left(\frac{1}{2}y^4\Big|_0^1\right) - \left(\frac{2}{3}y^3\Big|_0^1\right)^2 = \frac{1}{18}. \end{aligned}$$

And,

$$\begin{aligned} E[Var(Y|X)] &= E\left[\frac{(X-1)^2}{12}\right] = \int_{-\infty}^{\infty} \frac{(x-1)^2}{12}\, f_X(x)\,dx \\ &= \int_0^1 \frac{(x-1)^2}{12} \times 2(1-x)\,dx = \left(-\frac{1}{6}\right)\int_0^1 (x-1)^3\,dx = \left(-\frac{1}{6}\right)\left[\frac{1}{4}(x-1)^4\Big|_0^1\right] \\ &= \frac{1}{24} < \frac{1}{18} = Var(Y). \end{aligned}$$