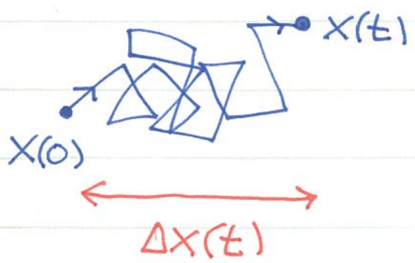
HH0049 Einstein relation in Brownian Motion.

Einstein was the first person to develop a quantitative theory for Brownian motions. Later, Perrin performed detail expt to verify the theory and won the Nobel Prize in 1926.



Later, we will derive the important relation

$$\boxed{(\Delta x)^2 = 2Dt}$$

where $(\Delta x)^2 = \langle x^2(t)\rangle$ since $\langle x(t)\rangle = 0$.

Diffusion from fluctuations:

Starting from the definition, $(\Delta x)^2 = \langle x^2(t)\rangle = \int_0^t dt_1 \int_0^t dt_2 \, \langle v(t_1) v(t_2)\rangle$

We're given $(\Delta x)^2 = 2Dt$ in the long-time limit,

$$D = \lim_{t\to\infty} \frac{1}{2}\frac{d(\Delta x)^2}{dt} = \lim_{t\to\infty} \int_0^t dt_2 \, \langle v(t) v(t_2)\rangle$$

velocity correlation fn.

Making use of translational invariance in time,

$\langle v(t) v(t_2)\rangle = \langle v(t-t_2) v(0)\rangle$ and change variable $t' = t - t_2$

$$\rightarrow \boxed{D = \lim_{t\to\infty} \int_0^t dt' \, \langle v(t') v(0)\rangle}$$

long-time limit expression for D !!

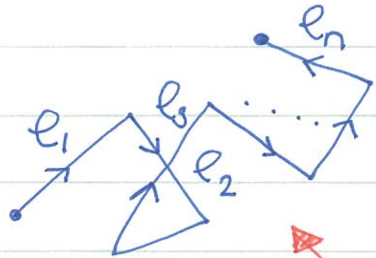
Within relaxation time approximation $\langle v(t') v(0)\rangle = \langle v^2\rangle e^{-t'/\tau_c}$

$$D = \int_0^\infty dt' \, \langle v^2\rangle \cdot e^{-t'/\tau_c} = \langle v^2\rangle \cdot \tau_c = \frac{T\tau_c}{M}$$

YES

Fractal dimension:

Consider the total length of the path at time t with observation time interval τ. The path is separated into $n = t/\tau$ segments



$$\boxed{L_\tau = \ell_1 + \ell_2 + \cdots + \ell_n}$$

$$\ell_i = \sqrt{(\Delta x_i)^2}$$

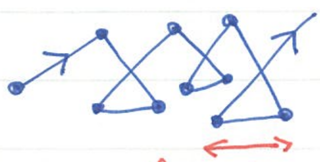
For $\tau \gg \tau_c$, each segment is described by $(\Delta x_i)^2 = 2D\tau$

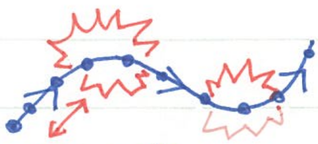
$$L_\tau = \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2} = n\sqrt{(2D\tau)} = \left[\sqrt{2D}\cdot t\right]\cdot\frac{1}{\sqrt{\tau}} \propto \tau^{1-d_f}$$

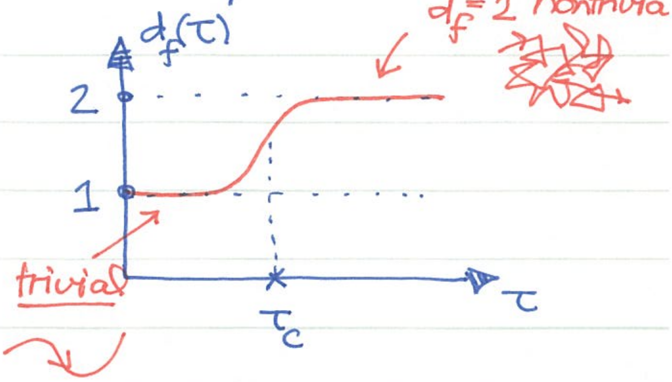
Thus, the fractal dimension $d_f = 2$. On the other hand, for $\tau \ll \tau_c$, the motion is ballistic $\ell_i \approx \bar{c}\cdot\tau$.

$$L_\tau = \sum_{i=1}^{n} \ell_i \approx n\bar{c}\cdot\tau = \bar{c}\,t \longrightarrow d_f = 1 \text{ as ordinary curve.}$$

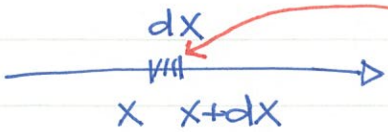
The difference can be understood this way:





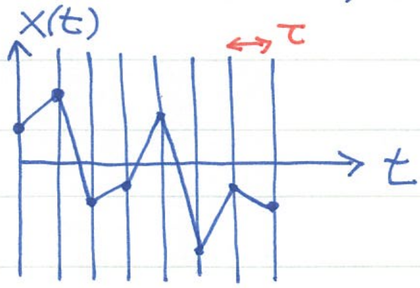


② Probability distribution: Now we want to derive the probability distribution function at microscopic level. Introduce $P(x,t)$:



$$\boxed{P(x)\,dx = \text{prob. to find particle in } dx}$$

For convenience, let's make time discrete: $t = n\tau$ — will take $\tau \to 0$ later.



Write down the master equation —

$$\boxed{P(x, t+\tau) = \int_{-\infty}^{+\infty} d\Delta\, f_\tau(\Delta)\, P(x-\Delta, t),}$$

where $f_\tau(\Delta)$ is the probability density for the particle to move Δ within the time interval τ.

$$\longrightarrow \int_{-\infty}^{+\infty} d\Delta\, f_\tau(\Delta) = 1 \quad \text{and} \quad f_\tau(\Delta) = f_\tau(-\Delta)$$

no intrinsic drift.

Now we are ready to expand the master equation....

$$\cancel{P} + \frac{\partial P}{\partial t}\cdot\tau \cong \int_{-\infty}^{+\infty} d\Delta\, f_\tau(\Delta)\left[P - \Delta\frac{\partial P}{\partial x} + \frac{1}{2}\Delta^2\frac{\partial^2 P}{\partial x^2}\right]$$

$$= \cancel{P} - \cancelto{0}{\langle\Delta\rangle}\frac{\partial P}{\partial x} + \frac{1}{2}\langle\Delta^2\rangle\frac{\partial^2 P}{\partial x^2}$$

The master equation is greatly simplified if the higher order terms can be ignored,

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2} \quad \text{with} \quad \boxed{D = \lim_{\tau\to 0}\frac{1}{2\tau}\int_{-\infty}^{+\infty} d\Delta\, f_\tau(\Delta)\,\Delta^2}$$

diffusion equation.

short-time expression for D

Suppose $P(x,t=0)=\delta(x)$ ← particle at the origin

In the Fourier space, $P(k,t=0)=\int dx\, e^{-ikx}\delta(x) = 1$.

The Diffusion eq. is

$$\frac{\partial P(k,t)}{\partial t} = -Dk^2 P(k,t)$$

simple to solve.

The solution is

$$P(k,t) = P(k,0)\,e^{-Dk^2t} = e^{-Dk^2t}$$

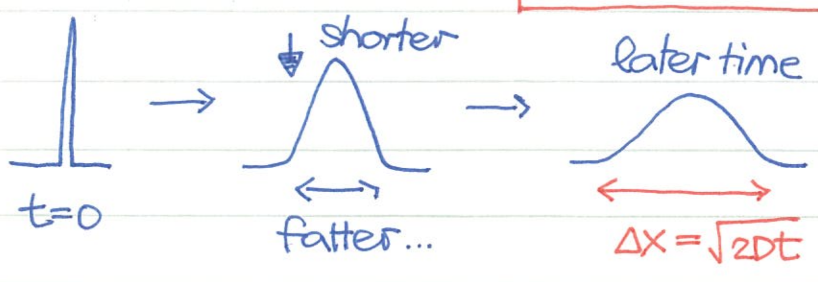
just Fourier transform it back to the coordinate space →

$$\boxed{P(x,t) = \frac{1}{\sqrt{4\pi Dt}}\,e^{-\frac{x^2}{4Dt}}}$$

The probability distribution for the Brownian motion is just the normal distribution. It is straightforward to compute the variance $\langle x^2\rangle$.

$$\langle x^2\rangle = 2Dt \longrightarrow \boxed{(\Delta x)^2 = 2Dt}$$

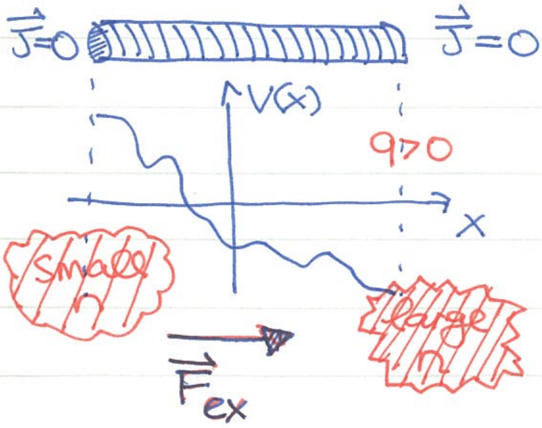
diffusion relation claimed previously.



Q: Why is $P(x,t)$ a normal distribution?

④ Einstein relation, at last... Now we would like to relate the diffusion constant D and the mobility $\tilde{\mu}$ again.

Consider an isolated wire with no current in or out. After thermal equilibrium is reached, the density dist.



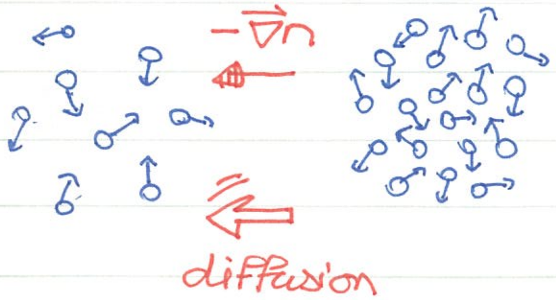
$$\boxed{n(\vec{r}) = n_0\, e^{-qV(\vec{r})/\tau}}$$

and the total current density is zero everywhere. There are two contributions to $\vec{J}_n$: ① external force $\vec{F}_{ex} = q\vec{E}$ ② density gradient $\vec{\nabla} n$.

The drift velocity is $\langle \vec{u} \rangle = \tilde{\mu}\, \vec{E} = -\tilde{\mu}\, \vec{\nabla} V(\vec{r})$ giving rise to a current density

$$\boxed{\vec{J}_n(\text{drift}) = n\langle \vec{u} \rangle = -n\tilde{\mu}\, \vec{\nabla} V}$$

On the other hand, the density gradient gives rise to counter propagating current, as described by Fick's law:



$$\boxed{\begin{aligned}\vec{J}_n(\text{diffusion}) &= -D\, \vec{\nabla} n \\ &= D\cdot\left(\frac{q}{\tau}\vec{\nabla} V\right)\cdot n\end{aligned}}$$

Combine both contributions together,

$$(\text{drift}) + (\text{diffusion}) = 0 \longrightarrow -\tilde{\mu}\, n\vec{\nabla} V + \frac{Dq}{\tau}\, n\vec{\nabla} V = 0$$

The detail balance for current leads to the Einstein relation,

$$\boxed{D = \frac{\tilde{\mu}}{q}\cdot \tau}$$

In some books, mobility is defined as $\langle \vec{u} \rangle = \tilde{\mu}\, \vec{F}_{ex}$, Einstein relation becomes $D = \tilde{\mu}\, \tau$ ☺

⑩ Brownian motion in stock market: Consider the index of stock market $S(t)$ where $t = 0, \tau, 2\tau, \cdots\cdots$ Introduce the (growth) ratio R_i defined as

$$R_1 = \frac{S(\tau)}{S(0)},\quad R_2 = \frac{S(2\tau)}{S(\tau)},\quad \cdots\cdots$$

At time $t = n\tau$, the index of the stock market is

$$S(n\tau) = \frac{S(n\tau)}{S((n-1)\tau)} \frac{S((n-1)\tau)}{S((n-2)\tau)} \cdots \frac{S(2\tau)}{S(\tau)} \frac{S(\tau)}{S(0)} \cdot S(0)$$

$$= R_n R_{n-1} \cdots R_2 R_1 \cdot S(0)$$, note that $0 < R_i < \infty$ are positive factors.

Introduce the random variable $X \equiv \log[S(n\tau)/S(0)]$.

$$\boxed{X = \sum_{i=1}^{n} \log R_i = x_1 + x_2 + \cdots + x_n}$$ where $x_i = \log R_i$

According to central limit theorem, the probability dist. for X is normal distribution if x_i are independent with the same (though unknown) distribution.

$$\boxed{P(x,n) = \frac{1}{\sqrt{2\pi\sigma^2 n}} e^{-\frac{(x-n\mu)^2}{2\sigma^2 n}}}$$

$\langle (\Delta x_i)^2 \rangle = \sigma^2$

$\langle x_i \rangle = \mu$

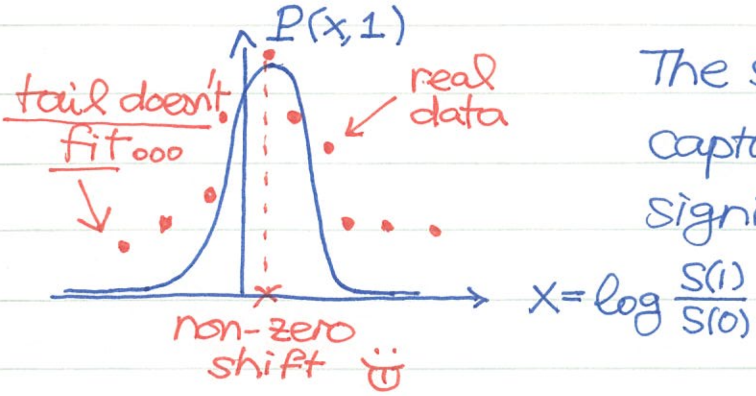
Compare with diffusion, $\sigma^2 n = 2Dt \rightarrow D = \sigma^2/2\tau$

The drift velocity $n\mu = \langle u \rangle t \rightarrow \langle u \rangle = \mu/\tau$

Rewrite the dist. fn as

$$P(x,t) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{(x-\langle u \rangle t)^2}{4Dt}}$$

$\log[S(t)/S(0)]$ moves like Brownian motion !!



The simple model more or less captures the main trend with significant tail corrections.

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