Linear Algebra

(20%) 1. Let $A = \begin{bmatrix} 1 & 3 & 0 & -1 & 2 \\ 0 & -2 & 4 & -2 & 0 \\ 3 & 11 & -4 & -1 & 6 \\ 2 & 5 & 3 & -4 & 0 \end{bmatrix}$.

(a) Find a basis for the row space of A. (5%)

(b) Find a basis for the column space of A. (5%)

(c) Find a basis for the null space of A. (10%)

(20%) 2. Let $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ -3 & 2 & -4 \end{bmatrix}$. Use the Gauss-Jordan method to determine whethe
A is invertible. If it is, find its inverse and express A as a product
a product of elementary matrices.

(10%) 3. Let $A = \begin{bmatrix} i & 2 \\ a & i \end{bmatrix}$.

(a) Find all values of the complex number a for which A is diagonalizabl
In this case, find a diagonal matrix D such that A is similar to D.(5

(b) Find all values of the complex number a for which A is unitarily diagonalizable. In this case, find a diagonal matrix C such that A is unitarily equivalent to C. (5%)

(10%) 4. If A is an $n \times n$ (real) matrix with the property that $\|Ax\| = 1$ for any unit vector x in $\mathbb{R}^n$, prove that A is an orthogonal matrix.

(20%) 5. (a) Prove that no invertible square matrix is nilpotent. (Recall that a matrix A is nilpotent if $A^k = 0$ for some positive integer k.) (5%)

(b) Determine all linear transformations from $\mathbb{R}$ to $\mathbb{R}$. Prove your asserti
($\mathbb{R}$ is the set of all real numbers.) (5%)

(c) Let $a \neq 0$ and b be vectors in $\mathbb{R}^n$. Find the projection of b on sp(a),
the subspace spanned by a. Prove your assertion. (5%)

(d) If A and B are $n \times n$ matrices satisfying AB = 0, prove that rank A + rank B $\leq$ n. (5%)

(20%) 6. (a) If A is a square matrix with

nullity $(T - 2I) = 4$, nullity $(T - 3I) = 5$,
nullity $(T - 2I)^2 = 6$, nullity $(T - 3I)^2 = 7$,
nullity $(T - 2I)^3 = 8$, nullity $(T - 3I)^3 = 7$,
nullity $(T - 2I)^4 = 9$,
nullity $(T - 2I)^5 = 10$,

use these data to find a Jordan canonical form of A. (10%)

(b) Find the algebraic and geometric multiplicities for every eigenvalue of A. (5%)

(c) Find the characteristic polynomial and the determinant of A. (5%)