＊作答前, 請先核對試題、答案卷（試卷）與准考證上之所組別與考試科目是否相符!!

1. Let A be an $m \times n$ matrix, and let $\mathbf{b}$ be a column vector such that $A\mathbf{x} = \mathbf{b}$ has a unique solution.

(i) (4%) Prove that $m \geq n$.

(ii) (3%) If $m = n$, must the system $A\mathbf{x} = \mathbf{b}$ always have a solution for every choice of $\mathbf{b}$? Either prove or produce a counter-example.

(iii) (3%) Answer part (ii) for the case where $m > n$.

2. (10%) Let $< \cdot, \cdot >$ be an inner product on $\mathbb{R}^n$. If $A \in M_{n\times n}(\mathbb{R})$ is such that $< Av, Av > = < v, v >$ for all $v \in \mathbb{R}^n$, is A necessarily an orthogonal matrix? Either prove or produce a counter-example.

3. Let $A \in M_{n\times n}(\mathbb{C})$ be a Hermitian matrix ($A^* = A$).

(i) (5%) Let $v \in \mathbb{C}^n$. Show that $A^k v = 0$ for some $k > 1$ implies $Av = 0$.

(ii) (5%) Let λ be a characteristic root (eigenvalue) of A. Show that $W_\lambda = E_\lambda$, where $W_\lambda := \{v \in \mathbb{C}^n \mid (A - \lambda I)^k v = 0 \text{ for some } k \geq 1\}$ is the generalized eigenspace of A corresponding to λ, $E_\lambda := \{v \in \mathbb{C}^n \mid (A - \lambda I)v = 0\}$ is the eigenspace of A, corresponding to λ.

4. (10%) Let A be an $m \times n$ matrix. Prove that $\text{rank}(AA^T) = \text{rank}(A^T A) = \text{rank}(A)$. ($A^T$ is the transpose of A.)

5. (10%) If $A \in M_{3\times 3}(\mathbb{R})$ satisfies $A^3 = 0$, show that there is an invertible $C \in M_{3\times 3}(\mathbb{R})$ such that $C^{-1}AC$ is an upper triangular matrix. What is the diagonal of $C^{-1}AC$?

6. Let V be a vector space over $\mathbb{R}$. Let $T : V \longrightarrow V$ be a linear transformation and $I : V \longrightarrow V$ the identity.

(i) (5%) Show that $\text{Ker}(T - I) \bigcap \text{Ker}(\text{T} + \text{I}) = \{0\}$.

(ii) (5%) Show that $T^2 = I$ if and only if V and $\text{Ker}(T - I) \oplus \text{Ker}(\text{T} + \text{I})$ are isomorphic. (Note: Ker = kernel)

7. (10%) Let A and B be two 3×3 matrices given by

$$A = \begin{bmatrix} 8 & 4 & -4 \\ 4 & 8 & 4 \\ -4 & 4 & 8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & 6 & -6 \\ 6 & 9 & -3 \\ -6 & -3 & 9 \end{bmatrix}.$$

Determine whether the two matrices A and B can represent the same linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ with respect to different pairs of ordered bases. (Answer with your computations.)

8. (10%) Let V be a finite-dimensional vector space over $\mathbb{R}$. Let $T : V \longrightarrow V$ be a linear transformation. Suppose W is a linear subspace of V, which is mapped into itself by T. Show that if V has a basis consisting of characteristic vectors (eigenvectors) of T, then W has a basis consisting of characteristic vectors of the restriction $T|_W$, of T to W.

9. Let U and V be two vector spaces over $\mathbb{R}$. Let $T : U \longrightarrow V$ be a linear transformation.

(i) (5%) Prove that $\mathrm{Ker}(T) = \{0\}$ if and only if T is one-to-one.

(ii) (5%) If T is one-to-one, show that the inverse map $T^{-1} : T(U) \longrightarrow U$ can be defined and T^{-1} is a linear transformation.

10. Let $\mathcal{P}_2(\mathbb{R})$ be the inner product space of all polynomials in x of degree ≤ 2 with real coefficients; for $p,\, q \in \mathcal{P}_2(\mathbb{R})$, $< p, q >_0 := \int_{-1}^{1} p(x)q(x)dx$.

(i) (5%) Find an orthonormal basis for $\mathcal{P}_2(\mathbb{R})$.

(ii) (5%) Find an isomorphism T from $\mathcal{P}_2(\mathbb{R})$ into $\mathbb{R}^3$ such that for all $p,\, q \in \mathcal{P}_2(\mathbb{R})$, $< p, q >_0 = < T(p), T(q) >$, where $< \cdot, \cdot >$ is the standard inner product on $\mathbb{R}^3$.