科目：工程數學B(3004)　　校系所組：中央大學通訊工程學系（甲組、乙組）
中央大學電機工程學系(系統與生醫組)
交通大學電子研究所(甲組、乙A組、乙B組)
交通大學電控工程研究所(甲組、乙組)
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清華大學電機工程學系(乙組、丙組、丁組)
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1. (15%) Let V be the set of all 2-by-2 real symmetric matrices, i.e.,

$$\mathsf{V} = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} : a, b, c \in \mathbb{R}, \text{ where } \mathbb{R} \text{ the set of real numbers} \right\}$$

and let W be the set of all 2-by-2 real matrices. Let $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ be a linear transformation defined by

$$\mathsf{T}\left(\begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) = \begin{pmatrix} a-c & -b+c \\ -2a+b+c & -a+c \end{pmatrix}, \quad \text{for all } a, b, c \in \mathbb{R}.$$

(a) (5%) Find a basis γ for the range space of T. (Please show details of your derivation.)

(b) (5%) Consider the Frobenius inner product where the inner product $\langle A, B \rangle$ between

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in \mathsf{W}$$

is defined by

$$\langle A, B \rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}.$$

Find an orthonormal basis α for the range space of T using the Gram-Schmidt process on the basis γ obtained in (a). (Please show details of your derivation.)

(c) (5%) Let

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

be an ordered basis for V and let α be the orthonormal basis obtained in (b). Find the matrix representation of T in the ordered basis β and α, i.e., $[\mathsf{T}]_\beta^\alpha$.

2. (10%) Suppose that

$$\det\left(\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \right) = -5$$

and $b_1c_2 = -b_2c_1 = 1$. Compute the determinant of the matrix

$$M = \begin{pmatrix} b_1+2c_1 & b_2+2c_2 & b_3+2c_3 \\ a_1 & a_2 & a_3+2 \\ 3b_1+c_1 & 3b_2+c_2 & 3b_3+c_3 \end{pmatrix}.$$

(Please show details of your derivation.)

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參考用

3. (15%) Let $\mathsf{V} = \mathsf{M}_{2\times2}(\mathbb{R})$ be the vector space consisting of all 2×2 matrices with real-valued entries and let T be a linear operator on V such that

$$\mathsf{T}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} c & d \\ a & b \end{pmatrix}.$$

Determine the eigenvalues of T and find an ordered basis β for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

4. (10%) Let

$$A = \begin{pmatrix} 2 & 3-3i \\ 3+3i & 5 \end{pmatrix}.$$

Find a unitary matrix P and a diagonal matrix D such that $P^*AP = D$.

5. [Fair coin tossing] (9%) Assume that a fair coin ($p = 1/2$) has been tossed over and over again, and results are independent each time.

(a) (3%) What is the probability p_1 that the "head" appears 5 times during the first 7 tosses?

(b) (3%) What is the probability p_2 that the 5^{th} head appears at the 7^{th} toss?

(c) (3%) Note that $p_2/p_1 = 5/7$. Prove that, in general, the probability that the kth "head" appears at the nth toss is k/n of the probability that the head appears k times during the first n tosses, for any positive integer $k < n$.

6. [Poisson random process] (10%) There are many volcanoes in the world. Assume that, in average, the number of volcano eruptions is 40 per 100 years. Also, assume that volcanoes erupt independently. Let us model the number $N(t)$ of volcano eruptions as a Poisson random process.

(a) (2%) What is the probability that $N(t) = 0$ during the next 10 years?

(b) (3%) Approximately, what is the probability that there will be two eruptions in March 2012?

(c) (5%) Start counting from now on, let X_3 be the time when the third eruption happens around the world. Write down the probability density function of X_3 and make a reasonable sketch.

注意：背面有試題

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7. **[True of False] (6%)** Please indicate TRUE or FALSE for each of the following statements.

(a) (1%) Let Ω be the sample space, and assume that $A_1 \cup A_2 = \Omega$, $A_1 \neq \emptyset$, and $A_2 \neq \emptyset$. Then, for any $B \subseteq \Omega$, we have $P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2)$.

(b) (1%) Let A, B be two events with nonzero probabilities. If $AB = \emptyset$ then A and B are not independent.

(c) (1%) Let X and Y be two random variables. Then $E(X+Y) = EX + EY$ only if X and Y are independent.

(d) (1%) Continued from above, $E(XY) = EX \cdot EY$ if X and Y are independent.

(e) (1%) Let X be a discrete random variable. If $E(X^4) = \left(E(X^2)\right)^2$, then $|X|$ is a constant.

(f) (1%) Let X be a random variable and c be a constant. If $c < E(X)$, then $P(X > c) \geq P(X \leq c)$.

8. **[Function of a Random Variable] (8%)** Let X be a uniformly distributed random variable on $(-2, 2)$. Let $Y = 4X^2$. Please find the cumulative distribution function of Y.

9. **[Moment Generating Function] (7%)** Let X be a random variable with a probability density function

$$f(x) = \begin{cases} \frac{3}{4}(1-x^2), & \text{if } -1 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Please find the moment generating function of X.

10. **[Central Limit Theorem] (10%)** Let $\Phi(x)$ be the cumulative distribution function of a standard normal distribution, i.e.,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt.$$

Let $X_1, X_2, \ldots, X_{20}$ be 20 independent and identically distributed random variables with a common probability density function

$$f(x) = \begin{cases} \frac{81}{x^4}, & \text{if } x > 3 \\ 0, & \text{if } x \leq 3. \end{cases}$$

Please use $\Phi(x)$ to calculate an approximation to $P\left(\sum_{i=1}^{20} X_i > 120\right)$.