國 立 清 華 大 學 命 題 紙

八十八學年度 數學系 系（所） 應用數學 組碩士班研究生招生考試

科目 線性代數 科號 0202 共 1 頁第 1 頁 *請在試卷【答案卷】內作答

1. (10%) If T is a linear transformation on $\mathbf{R}^2$, show that there exists $a, b, c, d \in \mathbf{R}$ such that $T(x, y) = (ax + by, cx + dy)$ for all $x, y \in \mathbf{R}$.

2. (15%) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear map such that $R(T) = N(T)$, where $R(T) = \{Tx \mid x \in \mathbf{R}^2\}$ and $N(T) = \{x \in \mathbf{R}^2 \mid Tx = 0\}$. Find $\operatorname{tr}(T)$ and $\det(T)$.

3. (20%) Let D_n be the determinant of the 1, 1, 1 tridiagonal $n \times n$ matrix

$$\begin{pmatrix} 1 & 1 & & & \\ 1 & 1 & 1 & & \\ & 1 & 1 & 1 & \\ & & \cdot & \cdot & \cdot \\ & & & 1 & 1 \end{pmatrix}.$$

Evaluate the value of D_n for each positive integer n.

4. (15%) Let $T : \mathbf{R}^4 \to \mathbf{R}^4$ be defined by $T(x, y, u, v) = (x + y, y - u, x + u, x + v)$. Let W be the T-cyclic subspace of $\mathbf{R}^4$ generated by $e_1 = (1, 0, 0, 0)$. Denote by T_W the restriction of T on W. Find $\operatorname{tr}(T_W)$ and $\det(T_W)$.

5. (15%) Let $A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$.

   (a) Find the minimal polynomial of A.

   (b) Find A^m, where m is a positive integer.

   (c) Evaluate $\lim_{m \to \infty} \operatorname{tr}(A^m)$.

6. (10%) Find $x \in \mathbf{R}^3$ such that $\|Ax - b\|^2$ is minimum, where

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}.$$

7. (15%) Let A be a 3×2 matrix and B be a 2×3 matrix such that $AB = \begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix}$.

   (a) Find the rank of A and the rank of B.

   (b) Show that there exists a 2×3 matrix C such that $CA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.