科目：線性代數(1002)

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1. (15 points) Let $T : V \to V$ be a linear transformation of the vector space V. Let $\lambda_1, \dots, \lambda_k$ be pairwise distinct eigenvalues of T; let v_i denote an eigenvector for λ_i, $1 \le i \le k$. Prove that $v_1, \dots, v_k$ are linearly independent.

2. (15 points) Determine all matrices A over all rational numbers Q that have distinct eigenvalues and satisfy $A^2 = 3A - 2I$.

3. (15 points) Let A be a square matrix. Prove that there is a diagonal matrix D whose diagonal entries are either $+1$ or -1 such that $\det(A + D) \neq 0$.

4. (15 points) In this problem, all matrices are viewed over the complex numbers. Let

$$A = \begin{bmatrix} 1 & 2 \\ -2 & x \end{bmatrix}.$$

For which complex numbers x, if any, is the matrix A not similar to a diagonal matrix? Justify your answer.

5. (20 points) Let A be an $n \times n$ matrix over the complex numbers and assume that the rank of A is equal to 1.
(a) What are the possible Jordan canonical forms for A? Justify your answer. (10 points)
(b) For each of the forms obtained in part (a), compute the characteristic polynomial of A and the minimal polynomial of A. (10 points)

6. (20 points) Let V be a finite dimensional vector space over a field F and $|V| \ge 2$. Let $T : V \to V$ be a linear transformation. If there exists a vector $v \in V$ such that V is spanned by $v, T(v), T^2(v), T^3(v), \dots$, prove that the characteristic polynomial of T is equal to the minimal polynomial of T.