Solution to Homework Assignment No. 1

1. (a) Perform elimination as follows:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -7 & -6 & -10 \\ 0 & -1 & -1 & -2 \end{array}\right] \begin{array}{l} \\ \text{(subtract } 3 \times \text{row 1)} \\ \text{(subtract } 2 \times \text{row 1)} \end{array}$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -7 & -6 & -10 \\ 0 & 0 & -\frac{1}{7} & -\frac{4}{7} \end{array}\right] \begin{array}{l} \\ \\ \text{(subtract } 1/7 \times \text{row 2)} \end{array}$$

This system is equivalent to

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -6 \\ 0 & 0 & -\frac{1}{7} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -10 \\ -\frac{4}{7} \end{bmatrix}.$$

Then we can solve the equations by back substitution as

$$\left\{\begin{array}{l} x + 2y + z = 3 \\ -7y - 6z = -10 \\ -\frac{1}{7}z = -\frac{4}{7} \end{array}\right. \Rightarrow \left\{\begin{array}{l} x = 3 - 2y - z \\ -7y = -10 + 6z \\ z = 4 \end{array}\right. \Rightarrow \left\{\begin{array}{l} x = 3 \\ y = -2 \\ z = 4. \end{array}\right.$$

The pivots are 1, -7, and $-1/7$, and the solution is $(x, y, z) = (3, -2, 4)$.

(b) Perform elimination as follows:

$$\left[\begin{array}{cccc|c} 0 & -1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 6 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right] \text{(exchange row 1 and 2)}$$

$$\Longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & -4 & -13 \\ 0 & -2 & -5 & -1 & -15 \end{array}\right] \begin{array}{l} \\ \\ \text{(subtract } 2 \times \text{row 1)} \\ \text{(subtract } 3 \times \text{row 1)} \end{array}$$

$$\Longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -3 & -2 & -13 \\ 0 & 0 & -3 & -3 & -15 \end{array}\right] \begin{array}{l} \\ \\ \text{(add } 2 \times \text{row 2)} \\ \text{(subtract } 2 \times \text{row 2)} \end{array}$$

$$\Longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -3 & -2 & -13 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right] \begin{array}{l} \\ \\ \\ \text{(subtract row 3)} \end{array}$$

This system is equivalent to

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ -13 \\ -2 \end{bmatrix}.$$

Then we can solve the equations by back substitution as

$$\begin{cases} x+y+z+t=6 \\ -y-z+t=0 \\ -3z-2t=-13 \\ -t=-2 \end{cases} \Rightarrow \begin{cases} x=6-y-z-t \\ -y=z-t \\ -3z=-13+2t \\ t=2 \end{cases} \Rightarrow \begin{cases} x=2 \\ y=-1 \\ z=3 \\ t=2. \end{cases}$$

The pivots are 1, −1, −3, and −1, and the solution is $(x, y, z, t) = (2, -1, 3, 2)$.

2. Perform elimination as follows:

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \overset{\boldsymbol{E}_{21}}{\Longrightarrow} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad (\text{add } 1/2 \times \text{row } 1)$$

$$\overset{\boldsymbol{E}_{32}}{\Longrightarrow} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad (\text{add } 2/3 \times \text{row } 2)$$

$$\overset{\boldsymbol{E}_{43}}{\Longrightarrow} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix} \quad (\text{add } 3/4 \times \text{row } 3)$$

This process can be expressed by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}.$$

Therefore, we have

$$\boldsymbol{E}_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{3}{4} & 1 \end{bmatrix}, \ \boldsymbol{E}_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } \boldsymbol{E}_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Applying these three elimination steps to the identity matrix $\boldsymbol{I}$ yields

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \overset{\boldsymbol{E}_{21}}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{\boldsymbol{E}_{32}}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{\boldsymbol{E}_{43}}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{bmatrix} = \boldsymbol{E}_{43}\boldsymbol{E}_{32}\boldsymbol{E}_{21}.$$

3. (a) Using the Gauss-Jordan method, we can have

$$\left[\ \boldsymbol{A} \mid \boldsymbol{I}\ \right] = \left[\begin{array}{ccc|ccc} \color{red}{1} & -2 & 1 & 1 & 0 & 0 \\ 2 & -1 & -1 & 0 & 1 & 0 \\ -2 & -5 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & \color{red}{3} & -3 & -2 & 1 & 0 \\ 0 & -9 & 9 & 2 & 0 & 1 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 3 & -3 & -2 & 1 & 0 \\ 0 & 0 & \color{red}{0} & -4 & 3 & 1 \end{array}\right].$$

Since we cannot obtain three nonzero pivots, $\boldsymbol{A}^{-1}$ does not exist.

(b) Using the Gauss-Jordan method, we can have

$$
\begin{aligned}
\left[\, \boldsymbol{B} \mid \boldsymbol{I} \,\right] &= \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & -3 & 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \\
&\Longrightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \\
&\Longrightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \\
&\Longrightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 & -1 & 1 \end{array}\right] \\
&\Longrightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & -2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 0 & 6 & -1 & -2 & 2 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 3 & -1 & -1 & 1 \end{array}\right] \\
&\Longrightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & -2 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 5 & -1 & -2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 3 & -1 & -1 & 1 \end{array}\right] \\
&\Longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -7 & 2 & 3 & -2 \\ 0 & 1 & 0 & 0 & 5 & -1 & -2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 3 & -1 & -1 & 1 \end{array}\right] \\
&\Longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -7 & 2 & 3 & -2 \\ 0 & 1 & 0 & 0 & 5 & -1 & -2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 & 1 & 1 & -1 \end{array}\right] = \left[\, \boldsymbol{I} \mid \boldsymbol{B}^{-1} \,\right].
\end{aligned}
$$

The inverse is hence

$$
\boldsymbol{B}^{-1} = \begin{bmatrix} -7 & 2 & 3 & -2 \\ 5 & -1 & -2 & 1 \\ 1 & 0 & 0 & 1 \\ -3 & 1 & 1 & -1 \end{bmatrix}.
$$

4. Performing elimination, we can have

$$\boldsymbol{A} = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} \xrightarrow{\boldsymbol{E}_1} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix} \begin{matrix} \\ \text{(subtract row 1)} \\ \text{(subtract row 1)} \\ \text{(subtract row 1)} \end{matrix}$$

$$\xrightarrow{\boldsymbol{E}_2} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix} \begin{matrix} \\ \\ \text{(subtract row 2)} \\ \text{(subtract row 2)} \end{matrix}$$

$$\xrightarrow{\boldsymbol{E}_3} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix} \begin{matrix} \\ = \boldsymbol{U}. \\ \\ \text{(subtract row 3)} \end{matrix}$$

This procedure can be viewed as

$$\boldsymbol{E}_3\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A} = \boldsymbol{U}$$

where

$$\boldsymbol{E}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \ \boldsymbol{E}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \text{ and } \boldsymbol{E}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Recording the elimination steps and changing the signs of the off-diagonal elements, we can have

$$\boldsymbol{L} = \boldsymbol{E}_1^{-1}\boldsymbol{E}_2^{-1}\boldsymbol{E}_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

We can therefore obtain $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$ as

$$\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

For $\boldsymbol{A}$ to have four pivots, the four conditions are:

$$a \neq 0, \ a \neq b, \ b \neq c, \text{ and } c \neq d.$$

5. (a) Performing elimination, we can have

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{bmatrix} \xrightarrow{\boldsymbol{E}_{21}} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 5 & 18 & 30 \end{bmatrix} \xrightarrow{\boldsymbol{E}_{31}} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 3 & 5 \end{bmatrix} \xrightarrow{\boldsymbol{E}_{32}} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix} = \boldsymbol{U}.$$

This procedure can be viewed as

$$\boldsymbol{E}_{32}\boldsymbol{E}_{31}\boldsymbol{E}_{21}\boldsymbol{A} = \boldsymbol{U}$$

where

$$\boldsymbol{E}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \boldsymbol{E}_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}, \text{ and } \boldsymbol{E}_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Recording the elimination steps and changing the signs of the off-diagonal elements, we can have

$$\boldsymbol{L} = \boldsymbol{E}_{21}^{-1}\boldsymbol{E}_{31}^{-1}\boldsymbol{E}_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix}.$$

We also find that $\boldsymbol{U} = \boldsymbol{D}\boldsymbol{L}^T$ where

$$\boldsymbol{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

We can therefore obtain $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T$ as

$$\begin{bmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Performing elimination, we can have

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \xRightarrow{\boldsymbol{E}_{21}} \begin{bmatrix} a & b \\ 0 & d - \frac{b^2}{a} \end{bmatrix} = \boldsymbol{U}.$$

This procedure can be viewed as

$$\boldsymbol{E}_{21}\boldsymbol{A} = \boldsymbol{U}$$

where

$$\boldsymbol{E}_{21} = \begin{bmatrix} 1 & 0 \\ -\frac{b}{a} & 1 \end{bmatrix}.$$

We can have

$$\boldsymbol{L} = \boldsymbol{E}_{21}^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{b}{a} & 1 \end{bmatrix}.$$

We also find that $\boldsymbol{U} = \boldsymbol{D}\boldsymbol{L}^T$ where

$$\boldsymbol{D} = \begin{bmatrix} a & 0 \\ 0 & d - \frac{b^2}{a} \end{bmatrix}.$$

We can therefore obtain $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T$ as

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{b}{a} & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d - \frac{b^2}{a} \end{bmatrix} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix}.$$

6. (a) (Lower triangular case)

Suppose $\boldsymbol{L}$ is an $n \times n$ lower triangular matrix with unit diagonal. We can use the Gauss-Jordan method to check if it has a full set of n pivots, which implies the matrix is invertible. We only need to do the Gaussian part. It means that the required operations are only to subtract the ith row from the jth row for $i < j$. Therefore, we can have

$$\begin{aligned}
\left[\ \boldsymbol{L}\ \middle|\ \boldsymbol{I}\ \right] &= \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ l_{2,1} & 1 & \ddots & \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\ l_{n,1} & \cdots & l_{n,n-1} & 1 & 0 & \cdots & 0 & 1 \end{array}\right] \\
&\Longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots & l'_{2,1} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & l'_{n,1} & \cdots & l'_{n,n-1} & 1 \end{array}\right] = \left[\ \boldsymbol{I}\ \middle|\ \boldsymbol{L}^{-1}\ \right].
\end{aligned}$$

Because the matrix has a unit diagonal, it has n pivots and $\boldsymbol{L}^{-1}$ is lower triangular with unit diagonal. The upper triangular case can be proved similarly.

(b) (Lower triangular case)

Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are two $n \times n$ lower triangular matrices with unit diagonal. We have $A_{i,j} = 0$ if $i < j$ and $A_{i,j} = 1$ if $i = j$, and $B_{i,j} = 0$ if $i < j$ and $B_{i,j} = 1$ if $i = j$. For $1 \leq i < j \leq n$, we have

$$\begin{aligned}
(AB)_{i,j} &= \sum_{k=1}^{n} A_{i,k} B_{k,j} \\
&= \sum_{k=1}^{j-1} A_{i,k} B_{k,j} + \sum_{k=j}^{n} A_{i,k} B_{k,j} \\
&= 0 + 0 \ (B_{k,j} = 0 \text{ when } k < j, \text{ and } A_{i,k} = 0 \text{ when } i < j \leq k.) \\
&= 0.
\end{aligned}$$

Therefore, $\boldsymbol{AB}$ is lower triangular. For $1 \leq i = j \leq n$, we have

$$\begin{aligned}
(AB)_{i,i} &= \sum_{k=1}^{n} A_{i,k} B_{k,i} \\
&= \sum_{k=1}^{i-1} A_{i,k} B_{k,i} + A_{i,i} B_{i,i} + \sum_{k=i+1}^{n} A_{i,k} B_{k,i} \\
&= 0 + 1 \cdot 1 + 0 \ (B_{k,i} = 0 \text{ when } k < i,\ A_{i,i} = B_{i,i} = 1, \text{ and } A_{i,k} = 0 \text{ when } i < k) \\
&= 1.
\end{aligned}$$

Therefore, $\boldsymbol{AB}$ has a unit diagonal. We can conclude that $\boldsymbol{AB}$ is also lower triangular with unit diagonal. The upper triangular case can be proved similarly.

(c) (Lower triangular case)
Let $\boldsymbol{L}$ be an $n \times n$ lower triangular matrix and $\boldsymbol{D}$ be a diagonal matrix with diagonal elements d_1, d_2,..., d_n. We can have

$$\begin{aligned}
\boldsymbol{LD} &= \begin{bmatrix} l_{1,1} & 0 & \cdots & 0 \\ l_{2,1} & l_{2,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ l_{n,1} & \cdots & l_{n,n-1} & l_{n,n} \end{bmatrix} \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{bmatrix} \\
&= \begin{bmatrix} d_1 l_{1,1} & 0 & \cdots & 0 \\ d_1 l_{2,1} & d_2 l_{2,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ d_1 l_{n,1} & \cdots & d_{n-1} l_{n,n-1} & d_n l_{n,n} \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{DL} &= \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{bmatrix} \begin{bmatrix} l_{1,1} & 0 & \cdots & 0 \\ l_{2,1} & l_{2,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ l_{n,1} & \cdots & l_{n,n-1} & l_{n,n} \end{bmatrix} \\
&= \begin{bmatrix} d_1 l_{1,1} & 0 & \cdots & 0 \\ d_2 l_{2,1} & d_2 l_{2,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ d_n l_{n,1} & \cdots & d_n l_{n,n-1} & d_n l_{n,n} \end{bmatrix}.
\end{aligned}$$

Therefore, the product of a lower triangular matrix and a diagonal matrix is still a lower triangular matrix. The upper triangular case can be proved similarly.

7. (a) (i) By **6**.(a), $\boldsymbol{L}_1^{-1}$ and $\boldsymbol{U}_2^{-1}$ both exist. Given $\boldsymbol{A} = \boldsymbol{L}_1\boldsymbol{D}_1\boldsymbol{U}_1$ and $\boldsymbol{A} = \boldsymbol{L}_2\boldsymbol{D}_2\boldsymbol{U}_2$, we can have

$$\begin{aligned}
& \boldsymbol{L}_2\boldsymbol{D}_2\boldsymbol{U}_2 = \boldsymbol{L}_1\boldsymbol{D}_1\boldsymbol{U}_1 \\
\Longrightarrow\ & \boldsymbol{L}_1^{-1}(\boldsymbol{L}_2\boldsymbol{D}_2\boldsymbol{U}_2)\boldsymbol{U}_2^{-1} = \boldsymbol{L}_1^{-1}(\boldsymbol{L}_1\boldsymbol{D}_1\boldsymbol{U}_1)\boldsymbol{U}_2^{-1} \\
\Longrightarrow\ & \boldsymbol{L}_1^{-1}\boldsymbol{L}_2\boldsymbol{D}_2 = \boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1}.
\end{aligned}$$

(ii) By **6**.(a), $\boldsymbol{L}_1^{-1}$ is lower triangular with unit diagonal. By **6**.(b), $\boldsymbol{L}_1^{-1}\boldsymbol{L}_2$ is lower triangular with unit diagonal. Therefore, by **6**.(c), $\boldsymbol{L}_1^{-1}\boldsymbol{L}_2\boldsymbol{D}_2$ is lower triangular. Similarly, $\boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1}$ is upper triangular.

(b) Let $\boldsymbol{M} = \boldsymbol{L}_1^{-1}\boldsymbol{L}_2\boldsymbol{D}_2 = \boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1}$. Then $\boldsymbol{M}$ is both lower and upper triangular, which implies that $\boldsymbol{M}$ is a diagonal matrix.

(i) Since $\boldsymbol{U}_1\boldsymbol{U}_2^{-1}$ has a unit diagonal, $\boldsymbol{M} = \boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1}$ has the same diagonal as $\boldsymbol{D}_1$. It implies that $\boldsymbol{M} = \boldsymbol{D}_1$. Similarly, we can have $\boldsymbol{M} = \boldsymbol{D}_2$. Therefore, $\boldsymbol{D}_1 = \boldsymbol{D}_2$.

(ii) For $\boldsymbol{M} = \boldsymbol{L}_1^{-1}\boldsymbol{L}_2\boldsymbol{D}_2 = \boldsymbol{D}_2$, we have $\boldsymbol{L}_1^{-1}\boldsymbol{L}_2 = \boldsymbol{I}$. Since the inverse matrix is unique, we have $\boldsymbol{L}_2 = (\boldsymbol{L}_1^{-1})^{-1} = \boldsymbol{L}_1$.

(iii) Similarly, for $\boldsymbol{M} = \boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1} = \boldsymbol{D}_1$, we have $\boldsymbol{U}_1\boldsymbol{U}_2^{-1} = \boldsymbol{I}$. It then implies that $\boldsymbol{U}_1 = (\boldsymbol{U}_2^{-1})^{-1} = \boldsymbol{U}_2$.

8. First do row exchange as

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix} \overset{\boldsymbol{P}_{31}}{\Longrightarrow} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 8 \\ 0 & 1 & 2 \end{bmatrix} = \boldsymbol{PA}$$

and then perform elimination as

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 8 \\ 0 & 1 & 2 \end{bmatrix} \overset{\boldsymbol{E}_{32}}{\Longrightarrow} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 8 \\ 0 & 0 & -\frac{2}{3} \end{bmatrix} = \boldsymbol{U}.$$

Then we have

$$\boldsymbol{E}_{32}(\boldsymbol{PA}) = \boldsymbol{U}$$

where

$$\boldsymbol{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \text{ and } \boldsymbol{E}_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix}.$$

We can have

$$\boldsymbol{L} = \boldsymbol{E}_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix}.$$

The factorization $\boldsymbol{PA} = \boldsymbol{LU}$ is hence given by

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 8 \\ 0 & 0 & -\frac{2}{3} \end{bmatrix}.$$

In order to factor $\boldsymbol{A}$ into $\boldsymbol{A} = \boldsymbol{L}_1\boldsymbol{P}_1\boldsymbol{U}_1$, we first perform elimination as

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix} \overset{\boldsymbol{E}_{21}}{\Longrightarrow} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix}$$

and then do row exchange as

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix} \overset{\boldsymbol{P}_{32}}{\Longrightarrow} \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \overset{\boldsymbol{P}_{21}}{\Longrightarrow} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \boldsymbol{U}_1.$$

Therefore,

$$\boldsymbol{U}_1 = \boldsymbol{P}_{21}\boldsymbol{P}_{32}\boldsymbol{E}_{21}\boldsymbol{A}$$

where

$$\boldsymbol{P}_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \boldsymbol{P}_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \text{ and } \boldsymbol{E}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiplying $\boldsymbol{E}_{21}^{-1}\boldsymbol{P}_{32}^{-1}\boldsymbol{P}_{21}^{-1}$ from the left to both sides, we can have

$$\boldsymbol{A} = \boldsymbol{E}_{21}^{-1}\boldsymbol{P}_{32}^{-1}\boldsymbol{P}_{21}^{-1}\boldsymbol{U}_1 = \boldsymbol{L}_1\boldsymbol{P}_1\boldsymbol{U}_1$$

where

$$\boldsymbol{P}_1 = \boldsymbol{P}_{32}^{-1}\boldsymbol{P}_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and

$$\boldsymbol{L}_1 = \boldsymbol{E}_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The factorization $\boldsymbol{A} = \boldsymbol{L}_1\boldsymbol{P}_1\boldsymbol{U}_1$ is hence given by

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$