Solution to Homework Assignment No. 2

1. (a) No, this subset is not a subspace of $\mathcal{R}^3$. Let $B_1 = \{(b_1, b_2, b_3) : b_1 b_2 b_3 = 0\}$. Consider $(1, 0, 0)$, $(0, 1, 1) \in B_1$. Since $(1, 0, 0) + (0, 1, 1) = (1, 1, 1) \notin B_1$, B_1 is not a subspace of $\mathcal{R}^3$.

(b) Yes, this subset is a subspace of $\mathcal{R}^3$. Let $B_2 = \{(b_1, b_2, b_3) : b_1 + b_2 + b_3 = 0\}$. Take two vectors $\boldsymbol{u} = (u_1, u_2, u_3) \in B_2$, $\boldsymbol{v} = (v_1, v_2, v_3) \in B_2$, where $u_1 + u_2 + u_3 = 0$ and $v_1 + v_2 + v_3 = 0$. Then we need to check the following two conditions:

- Consider $\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$. Since $(u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3) = (u_1 + u_2 + u_3) + (v_1 + v_2 + v_3) = 0$, $\boldsymbol{u} + \boldsymbol{v} \in B_2$.
- For any $c \in \mathcal{R}$, consider $c\boldsymbol{u} = (cu_1, cu_2, cu_3)$. Since $cu_1 + cu_2 + cu_3 = c(u_1 + u_2 + u_3) = 0$, $c\boldsymbol{u} \in B_2$.

Therefore, B_2 is a subspace of $\mathcal{R}^3$.

(c) No, this subset is not a subspace. Let $B_3 = \{(b_1, b_2, b_3) : b_1 \le b_2 \le b_3\}$. Consider $(1, 2, 3) \in B_3$. Since $(-1)(1, 2, 3) = (-1, -2, -3) \notin B_3$, B_3 is not a subspace of $\mathcal{R}^3$.

2. (a) Yes, $S + T$ is a subspace of V. Let $\boldsymbol{x}_1 = \boldsymbol{s}_1 + \boldsymbol{t}_1 \in S + T$ and $\boldsymbol{x}_2 = \boldsymbol{s}_2 + \boldsymbol{t}_2 \in S + T$, where $\boldsymbol{s}_1, \boldsymbol{s}_2 \in S$ and $\boldsymbol{t}_1, \boldsymbol{t}_2 \in T$. We need to check the following two conditions:

- Consider $\boldsymbol{x}_1 + \boldsymbol{x}_2 = (\boldsymbol{s}_1 + \boldsymbol{t}_1) + (\boldsymbol{s}_1 + \boldsymbol{t}_1) = (\boldsymbol{s}_1 + \boldsymbol{s}_2) + (\boldsymbol{t}_1 + \boldsymbol{t}_2)$. Since $\boldsymbol{s}_1, \boldsymbol{s}_2 \in S$ and S is a subspace, we have $\boldsymbol{s}_1 + \boldsymbol{s}_2 \in S$. Similarly, we have $\boldsymbol{t}_1 + \boldsymbol{t}_2 \in T$. Therefore, $\boldsymbol{x}_1 + \boldsymbol{x}_2 = (\boldsymbol{s}_1 + \boldsymbol{s}_2) + (\boldsymbol{t}_1 + \boldsymbol{t}_2) \in S + T$.
- For any $c \in \mathcal{R}$, consider $c\boldsymbol{x}_1 = c\boldsymbol{s}_1 + c\boldsymbol{t}_1$. Since $\boldsymbol{s}_1 \in S$, $\boldsymbol{t}_1 \in T$ and S, T are subspaces of V, we have $c\boldsymbol{s}_1 \in S$ and $c\boldsymbol{t}_1 \in T$. Therefore, $c\boldsymbol{x}_1 = c\boldsymbol{s}_1 + c\boldsymbol{t}_1 \in S + T$.

Therefore, $S + T$ is a subspace of the vector space V.

(b) No, $S \cup T$ is in general not a subspace of V. Consider $S = \{(x, 0) : x \in \mathcal{R}\}$, $T = \{(0, y) : y \in \mathcal{R}\}$ are two subspaces of $\mathcal{R}^2$. Take $(1, 0) \in S$, $(0, 1) \in T$, and hence $(1, 0) \in S \cup T$, $(0, 1) \in S \cup T$. Since $(1, 0) + (0, 1) = (1, 1) \notin S \cup T$, $S \cup T$ is not a subspace.

3. (a) Reduce the matrix $\boldsymbol{A}$ to the reduced row echelon (RRE) form:

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} = \boldsymbol{R}_A$$

$$\Rightarrow \begin{cases} x_1 \quad\quad - x_3 = 0 \\ \quad\quad + x_2 - x_3 = 0. \end{cases}$$

The pivot variables are x_1 and x_2, and the free variables are x_3 and x_4. Setting $x_3 = 1$, $x_4 = 0$, we have $x_1 = 1$ and $x_2 = 1$. Setting $x_3 = 0$, $x_4 = 1$, we have

$x_1 = 0$ and $x_2 = 0$. Therefore, the nullspace of matrix $\boldsymbol{A}$ can be given by

$$\mathcal{N}(\boldsymbol{A}) = \left\{ \boldsymbol{x} : \boldsymbol{x} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, x_3, x_4 \in \mathcal{R} \right\}.$$

(b) Reduce the matrix $\boldsymbol{B}$ to the RRE form:

$$\boldsymbol{B} = \begin{bmatrix} 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \boldsymbol{R}_B$$

$$\Rightarrow \left\{ \begin{array}{rrrcl} x_2 & & -x_4 & = & 0 \\ & +x_3 & -x_4 & = & 0. \end{array} \right.$$

The pivot variables are x_2 and x_3, and the free variables are x_1 and x_4. Setting $x_1 = 1$, $x_4 = 0$, we have $x_2 = 0$ and $x_3 = 0$. Setting $x_1 = 0$, $x_4 = 1$, we have $x_2 = 1$ and $x_3 = 1$. Therefore, the nullspace of matrix $\boldsymbol{B}$ can be given by

$$\mathcal{N}(\boldsymbol{B}) = \left\{ \boldsymbol{x} : \boldsymbol{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, x_1, x_4 \in \mathcal{R} \right\}.$$

(c) Reduce the matrix $\boldsymbol{C}$ to the RRE form:

$$\boldsymbol{C} = \begin{bmatrix} \mathcal{A} \\ \mathcal{B} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \boldsymbol{R}_C \Longrightarrow \left\{ \begin{array}{rrrrcl} x_1 & & & -x_4 & = & 0 \\ & x_2 & & -x_4 & = & 0 \\ & & x_3 & -x_4 & = & 0. \end{array} \right.$$

The pivot variables are x_1, x_2 and x_3, and the free variable is x_4. Setting $x_4 = 1$, we have $x_1 = 1$, $x_2 = 1$ and $x_3 = 1$. Therefore, the nullspace of matrix $\boldsymbol{C}$ can be given by

$$\mathcal{N}(\boldsymbol{C}) = \left\{ \boldsymbol{x} : \boldsymbol{x} = x_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, x_4 \in \mathcal{R} \right\}.$$

4. (a) First, we perform Gaussian elimination:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & c-1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \boldsymbol{U}.$$

To find the pivots, we must check if $c - 1 = 0$. There are two cases:

- If $c-1=0$, i.e., $c=1$, we have

$$\boldsymbol{U}_1 = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \boldsymbol{R}_1$$

where $\boldsymbol{R}_1$ is the RRE form with rank 1.

- If $c-1 \neq 0$, i.e., $c \neq 1$, we can obtain

$$\boldsymbol{U}_2 = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \boldsymbol{R}_2$$

where $\boldsymbol{R}_2$ is the RRE form with rank 2.

(b) The matrix $\boldsymbol{B}$ is already an upper triangular matrix. To find the pivots, we need to check if $1-d=0$ and $2-d=0$. There are three cases:

- If $1-d=0$, i.e., $d=1$, we have

$$\boldsymbol{B}_1 = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \boldsymbol{R}_3$$

where $\boldsymbol{R}_3$ is the RRE form with rank 1.

- If $2-d=0$, i.e., $d=2$, we have

$$\boldsymbol{B}_2 = \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} = \boldsymbol{R}_4$$

where $\boldsymbol{R}_4$ is the RRE form with rank 1.

- If $1-d \neq 0$ and $2-d \neq 0$, i.e., $d \neq 1$ and $d \neq 2$, we have

$$\boldsymbol{B}_3 = \begin{bmatrix} 1-d & 2 \\ 0 & 2-d \end{bmatrix} \Rightarrow \begin{bmatrix} 1-d & 0 \\ 0 & 2-d \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \boldsymbol{R}_5$$

where $\boldsymbol{R}_5$ is the RRE form with rank 2.

5. To find the complete solution, we reduce the matrix to the RRE form:

$$\left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & 4 \\ 2 & 5 & 7 & 6 & 3 \\ 2 & 3 & 5 & 2 & 5 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & -2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 2 & 0 & 2 & -4 & 8 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_1 + x_3 - 2x_4 = 4 \\ x_2 + x_3 + 2x_4 = -1. \end{cases}$$

Thus, the pivot variables are x_1 and x_2, and the free variables are x_3 and x_4. Setting $x_3 = 0$ and $x_4 = 0$, we can obtain $x_1 = 4$ and $x_2 = -1$. Therefore, a particular solution can be given by

$$\boldsymbol{x}_p = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

To find the general soultion $\boldsymbol{x}_n$, we let

$$\begin{cases} x_1 \qquad + x_3 - 2x_4 = 0 \\ \qquad x_2 + x_3 + 2x_4 = 0. \end{cases}$$

Setting $x_3 = 1$, $x_4 = 0$, we have $x_1 = -1$, $x_2 = -1$. Setting $x_3 = 0$, $x_4 = 1$, we have $x_1 = 2$, $x_2 = -2$. Therefore, the general solution $\boldsymbol{x}_n$ is

$$\boldsymbol{x}_n = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

As a result, the complete solution to this problem is

$$\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

6. (a) Since $\boldsymbol{x}$ is a 2 by 1 vector and $\boldsymbol{Ax}$ is a 2 by 1 vector, $\boldsymbol{A}$ is a 2 by 2 matrix. Since $\boldsymbol{Ax} = \boldsymbol{0}$ has one special solution, we know that the rank of $\boldsymbol{A} = 2 - 1 = 1$. Let

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Since $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a special solution to $\boldsymbol{Ax} = \boldsymbol{0}$, we have

$$\boldsymbol{Ax} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which gives $\boldsymbol{A} = \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix}$. By applying the particular solution to $\boldsymbol{Ax} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, we can obtain

$$\boldsymbol{Ax} = \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \Longrightarrow \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Therefore, we find the matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$, which is clearly with rank 1.

(b) Since $\boldsymbol{x}$ is a 3 by 1 vector and $\boldsymbol{Bx}$ is a 2 by 1 vector, $\boldsymbol{B}$ is a 2 by 3 matrix. Hence the rank r of $\boldsymbol{B}$ is at most 2. We discuss the following two cases:

- If $r = 2$, i.e., $\boldsymbol{B}$ has full row rank, $\boldsymbol{Bx} = \boldsymbol{b}$ always has infinite solutions.
- If $r < 2$, $\boldsymbol{Bx} = \boldsymbol{b}$ has 0 or infinite solutions.

Therefore, we cannot find a matrix $\boldsymbol{B}$ with only one solution to $\boldsymbol{Bx} = \boldsymbol{b}$.

7. Yes, they are linearly independent. Consider the following equation:

$$\begin{aligned}&b_1(2\boldsymbol{w}_1+\boldsymbol{w}_2+\boldsymbol{w}_3)+b_2(\boldsymbol{w}_1+2\boldsymbol{w}_2+\boldsymbol{w}_3)+b_3(\boldsymbol{w}_1+\boldsymbol{w}_2+2\boldsymbol{w}_3)=0\\ &\Rightarrow (2b_1+b_2+b_3)\boldsymbol{w}_1+(b_1+2b_2+b_3)\boldsymbol{w}_2+(b_1+b_2+2b_3)\boldsymbol{w}_3=0.\end{aligned}$$

Since $\boldsymbol{w}_1$, $\boldsymbol{w}_2$, $\boldsymbol{w}_3$ are linearly independent, we know the only solution to the above equation is

$$\begin{cases} 2b_1 + b_2 + b_3 = 0\\ b_1 + 2b_2 + b_3 = 0\\ b_1 + b_2 + 2b_3 = 0.\end{cases}$$

Since the matrix $\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}$ is invertible, we have $b_1=b_2=b_3=0$. Therefore, $2\boldsymbol{w}_1+\boldsymbol{w}_2+\boldsymbol{w}_3$, $\boldsymbol{w}_1+2\boldsymbol{w}_2+\boldsymbol{w}_3$, $\boldsymbol{w}_1+\boldsymbol{w}_2+2\boldsymbol{w}_3$ are linearly independent.

8. (a) Take a matrix $\boldsymbol{m}\in M$. Since all the column sums are zero, we can assume

$$\boldsymbol{m}=\begin{bmatrix}a&b&c\\-a&-b&-c\end{bmatrix}$$

where $a,b,c\in\mathcal{R}$. Then we have

$$\begin{aligned}\boldsymbol{m}&=\begin{bmatrix}a&0&0\\-a&0&0\end{bmatrix}+\begin{bmatrix}0&b&0\\0&-b&0\end{bmatrix}+\begin{bmatrix}0&0&c\\0&0&-c\end{bmatrix}\\ &=a\begin{bmatrix}1&0&0\\-1&0&0\end{bmatrix}+b\begin{bmatrix}0&1&0\\0&-1&0\end{bmatrix}+c\begin{bmatrix}0&0&1\\0&0&-1\end{bmatrix}.\end{aligned}$$

Thus, $\begin{bmatrix}1&0&0\\-1&0&0\end{bmatrix}$, $\begin{bmatrix}0&1&0\\0&-1&0\end{bmatrix}$, $\begin{bmatrix}0&0&1\\0&0&-1\end{bmatrix}$ span the vector space M. Consider

$$\begin{aligned}&x\begin{bmatrix}1&0&0\\-1&0&0\end{bmatrix}+y\begin{bmatrix}0&1&0\\0&-1&0\end{bmatrix}+z\begin{bmatrix}0&0&1\\0&0&-1\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}\\ &\Longrightarrow \begin{bmatrix}x&y&z\\-x&-y&-z\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}.\end{aligned}$$

Then we have $x=y=z=0$, and hence $\begin{bmatrix}1&0&0\\-1&0&0\end{bmatrix}$, $\begin{bmatrix}0&1&0\\0&-1&0\end{bmatrix}$, $\begin{bmatrix}0&0&1\\0&0&-1\end{bmatrix}$ are linearly independent. As a result, $\begin{bmatrix}1&0&0\\-1&0&0\end{bmatrix}$, $\begin{bmatrix}0&1&0\\0&-1&0\end{bmatrix}$, $\begin{bmatrix}0&0&1\\0&0&-1\end{bmatrix}$ are a basis for M.

(b) Assume N is the subspace of M whose columns and rows both add to zero. Let $\boldsymbol{n}\in N$. By the property of N, we can assume

$$\boldsymbol{n}=\begin{bmatrix}a&b&-a-b\\-a&-b&a+b\end{bmatrix}$$

where $a, b \in \mathcal{R}$. Then we have

$$\begin{aligned}
\boldsymbol{n} &= \begin{bmatrix} a & 0 & -a \\ -a & 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b & -b \\ 0 & -b & b \end{bmatrix} \\
&= a \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}.
\end{aligned}$$

Thus, $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ span the subspace N. Consider

$$\begin{aligned}
& h \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix} + k \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\Longrightarrow & \begin{bmatrix} h & k & -h-k \\ -h & -k & h+k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

Then we have $h = k = 0$, and hence $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ are linearly independent. As a result, $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ are a basis for N.