EE 2030 Linear Algebra — Chi-chao Chao
Spring 2011

Homework Assignment No. 3
Due 10:10am, April 22, 2011

`Reading:` Strang, Sections 3.6–4.3.

`Problems for Solution:`

1. Find a basis for each of the four subspaces associated with

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

2. Three matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ satisfy

$$\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}.$$

   (a) Show that $\text{rank}(\boldsymbol{C}) \le \text{rank}(\boldsymbol{B})$. (*Hint:* Argue that the rows of $\boldsymbol{C}$ are linear combinations of the rows of $\boldsymbol{B}$.)

   (b) Also show that $\text{rank}(\boldsymbol{C}) \le \text{rank}(\boldsymbol{A})$. (*Hint:* $\boldsymbol{C}^T = \boldsymbol{B}^T\boldsymbol{A}^T$.)

3. (a) Suppose S is spanned by the vectors $(1, 2, 2, 3)$ and $(1, 3, 3, 2)$. Find two vectors that span $S^\perp$. (*Hint:* This is the same as solving $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ for which $\boldsymbol{A}$?)

   (b) If P is the plane of vectors in $\mathcal{R}^4$ satisfying $x_1 + x_2 + x_3 + x_4 = 0$, find a basis for $P^\perp$. (*Hint:* Construct a matrix that has P as its nullspace.)

4. Find a basis for the nullspace of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}$$

   and verify that it is orthogonal to the row space. Given $\boldsymbol{x} = (3, 3, 3)$, split it into a row space component $\boldsymbol{x}_r$ and a nullspace component $\boldsymbol{x}_n$.

5. Project $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$. Find the projection $\boldsymbol{p}$. Also find $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$. It should be orthogonal to the columns of $\boldsymbol{A}$.

   (a) $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$.

   (b) $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 4 \\ 6 \\ 4 \end{bmatrix}$.

6. (a) Find the projection matrix $\boldsymbol{P}_C$ onto the column space of $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{bmatrix} 2 & 4 & 4 \\ 5 & 10 & 10 \end{bmatrix}.$$

(*Hint:* Look closely at the matrix!)

(b) Find the 3 by 3 projection matrix $\boldsymbol{P}_R$ onto the row space of $\boldsymbol{A}$. Multiply $\boldsymbol{B} = \boldsymbol{P}_C \boldsymbol{A} \boldsymbol{P}_R$. Your answer $\boldsymbol{B}$ should be a little surprising—can you explain it?

7. We have four data points with $b = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$.

(a) Find the closest parabola $b = C_1 + D_1 t + E_1 t^2$ to the four points. The error vector $\boldsymbol{e}$ is defined as in class. What is $\|\boldsymbol{e}\|^2$ now?

(b) Find the closest cubic $b = C_2 + D_2 t + E_2 t^2 + F_2 t^3$ to the four points. What is $\|\boldsymbol{e}\|^2$ now?

8. Let

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

(a) Show that the partial derivatives of $\|\boldsymbol{Ax}\|^2$ with respect to $x_1, x_2, \ldots, x_n$ fill the vector $2\boldsymbol{A}^T\boldsymbol{Ax}$.

(b) Show that the partial derivatives of $2\boldsymbol{b}^T\boldsymbol{Ax}$ fill the vector $2\boldsymbol{A}^T\boldsymbol{b}$.

(c) Show that the partial derivatives of $\|\boldsymbol{Ax} - \boldsymbol{b}\|^2$ are zero when $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$.