Solution to Homework Assignment No. 4

1. Since the columns of $\boldsymbol{A}$ are independent, let $\boldsymbol{a}_1 = (1,1,0)^T$, $\boldsymbol{a}_2 = (1,0,1)^T$, and $\boldsymbol{a}_3 = (0,1,1)^T$. By the Gram-Schmidt process, we can have

$$\boldsymbol{A}_1 = \boldsymbol{a}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix} \implies \boldsymbol{q_1} = \frac{\boldsymbol{A_1}}{\|\boldsymbol{A_1}\|} = \begin{bmatrix}\sqrt{2}/2\\\sqrt{2}/2\\0\end{bmatrix}$$

$$\boldsymbol{A}_2 = \boldsymbol{a_2} - (\boldsymbol{q_1}^T\boldsymbol{a_2})\boldsymbol{q_1} = \begin{bmatrix}1/2\\-1/2\\1\end{bmatrix} \implies \boldsymbol{q_2} = \frac{\boldsymbol{A_2}}{\|\boldsymbol{A_2}\|} = \begin{bmatrix}\sqrt{6}/6\\-\sqrt{6}/6\\\sqrt{6}/3\end{bmatrix}$$

$$\boldsymbol{A}_3 = \boldsymbol{a_3} - (\boldsymbol{q_1}^T\boldsymbol{a_3})\boldsymbol{q_1} - (\boldsymbol{q_2}^T\boldsymbol{a_3})\boldsymbol{q_2} = \begin{bmatrix}-2/3\\2/3\\2/3\end{bmatrix} \implies \boldsymbol{q_3} = \frac{\boldsymbol{A_3}}{\|\boldsymbol{A_3}\|} = \begin{bmatrix}-\sqrt{3}/3\\\sqrt{3}/3\\\sqrt{3}/3\end{bmatrix}.$$

Therefore,

$$\boldsymbol{A} = \begin{bmatrix}\boldsymbol{q_1} & \boldsymbol{q_2} & \boldsymbol{q_3}\end{bmatrix}\begin{bmatrix}\boldsymbol{q_1}^T\boldsymbol{a}_1 & \boldsymbol{q_1}^T\boldsymbol{a}_2 & \boldsymbol{q_1}^T\boldsymbol{a}_3\\ 0 & \boldsymbol{q_2}^T\boldsymbol{a}_2 & \boldsymbol{q_2}^T\boldsymbol{a}_3\\ 0 & 0 & \boldsymbol{q_3}^T\boldsymbol{a}_3\end{bmatrix} = \boldsymbol{QR}$$

which gives

$$\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix} = \begin{bmatrix}\sqrt{2}/2 & \sqrt{6}/6 & -\sqrt{3}/3\\ \sqrt{2}/2 & -\sqrt{6}/6 & \sqrt{3}/3\\ 0 & \sqrt{6}/3 & \sqrt{3}/3\end{bmatrix}\begin{bmatrix}\sqrt{2} & \sqrt{2}/2 & \sqrt{2}/2\\ 0 & \sqrt{6}/2 & \sqrt{6}/6\\ 0 & 0 & 2\sqrt{3}/3\end{bmatrix}.$$

2. (a) Consider

$$\boldsymbol{A}^T = \begin{bmatrix}2&1&2\\1&1&1\end{bmatrix} \overset{\text{RRE}}{\Longrightarrow} \boldsymbol{R} = \begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}.$$

Since $(1,0,1)\cdot(0,1,0) = 0$ and $\{(1,0,1),(0,1,0)\}$ forms a basis of the column space of $\boldsymbol{A}$, we can obtain

$$\boldsymbol{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix} \text{ and } \boldsymbol{q}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}.$$

For $\boldsymbol{A}^T$, $(-1,0,1)$ is a special solution. That is to say, $(-1,0,1)$ is orthogonal to the column space of $\boldsymbol{A}$. Since $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ span the column space of $\boldsymbol{A}$, we can choose

$$\boldsymbol{q}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}.$$

(b) Since $\boldsymbol{q}_3$ is a special solution to $\boldsymbol{A}^T\boldsymbol{x} = \boldsymbol{0}$, the left nullspace of $\boldsymbol{A}$ contains $\boldsymbol{q}_3$.

(c) Form (a), we have

$$\begin{aligned} \boldsymbol{A} &= \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1^T \boldsymbol{a}_1 & \boldsymbol{q}_1^T \boldsymbol{a}_2 \\ 0 & \boldsymbol{q}_2^T \boldsymbol{a}_2 \end{bmatrix} \\ &= \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix} \\ &= \boldsymbol{QR}. \end{aligned}$$

Then we can obtain the solution

$$\hat{\boldsymbol{x}} = \boldsymbol{R}^{-1} \boldsymbol{Q}^T \boldsymbol{b} = \begin{bmatrix} 1/\sqrt{2} & -1 \\ -1/\sqrt{2} & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix} = \begin{bmatrix} 9 \\ -3 \end{bmatrix}.$$

3. Since

$$\int_{-1}^{1} 1 \cdot x dx = \int_{-1}^{1} x dx = (x^2/2)\Big|_{x=-1}^{x=1} = 0$$

$$\int_{-1}^{1} 1 \cdot [x^2 - (1/3)] dx = \int_{-1}^{1} [x^2 - (1/3)] dx = [x^3/3 - (1/3)x]\Big|_{x=-1}^{x=1} = 0$$

$$\int_{-1}^{1} x \cdot [x^2 - (1/3)] dx = \int_{-1}^{1} [x^3 - (1/3)x] dx = [x^4/4 - (1/6)x^2]\Big|_{x=-1}^{x=1} = 0$$

we know that 1, x, and $x^2 - (1/3)$ are orthogonal, when the integration is from $x = -1$ to $x = 1$. Furthermore, $f(x) = 2x^2 = (2/3) \cdot 1 + 0 \cdot x + 2 \cdot [x^2 - (1/3)]$.

4. For the first matrix, doing Gaussian elimination, we have

$$\begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -10 & -20 & -30 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Its determinate is equal to $1 \cdot (-10) \cdot 0 \cdot 0 = 0$.

For the second matrix, doing Gaussian elimination, we have

$$\begin{bmatrix} 1 & t & t^2 & t^3 \\ t & 1 & t & t^2 \\ t^2 & t & 1 & t \\ t^3 & t^2 & t & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & t & t^2 & t^3 \\ 0 & 1-t^2 & t-t^3 & t^2-t^4 \\ 0 & 0 & 1-t^2 & t-t^3 \\ 0 & 0 & 0 & 1-t^2 \end{bmatrix}.$$

Its determinate is equal to $1 \cdot (1-t^2) \cdot (1-t^2) \cdot (1-t^2) = (1-t^2)^3$.

5. For the big formula, the determinant of $\boldsymbol{A}$ is the sum of $5! = 120$ simple determinants, times 1 or -1, and every simple determinant chooses one entry from each row and column. If some simple determinant of $\boldsymbol{A}$ avoids all the zero entries in $\boldsymbol{A}$, then it cannot choose one entry from each column. Thus every simple determinant of $\boldsymbol{A}$ must choose at least one zero entry, and hence all 120 terms are zero in the big formula for $\det \boldsymbol{A}$. That is to say, the determinant of this matrix is zero.

6. Let $D_n = |\boldsymbol{A_n}|$ where $\boldsymbol{A_n}$ is an n by n matrix. For $n \geq 3$, we have

$$D_n = \begin{vmatrix} 1 & -1 \;\; 0 \;\; \cdots \;\; 0 \\ 1 & \\ 0 & \boldsymbol{A_{n-1}} \\ \vdots & \\ 0 & \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 0\cdots & 0 \\ 1 & 1 & -1 & 0\cdots & 0 \\ 0 & 1 & & & \\ 0 & 0 & & \boldsymbol{A_{n-2}} & \\ \vdots & \vdots & & & \\ 0 & 0 & & & \end{vmatrix}.$$

Applying the cofactor formula to the first row, we can have

$$\begin{aligned} D_n &= 1 \cdot (-1)^{1+1} |\boldsymbol{A_{n-1}}| + (-1) \cdot (-1)^{1+2} \begin{vmatrix} 1 & -1 \;\; 0 \;\; \cdots \;\; 0 \\ 0 & \\ 0 & \boldsymbol{A_{n-2}} \\ \vdots & \\ 0 & \end{vmatrix} \\ &= D_{n-1} + 1 \cdot (-1)^{1+1} |\boldsymbol{A_{n-2}}| \ \ \text{(apply the cofactor formula to the first column)} \\ &= D_{n-1} + D_{n-2}. \end{aligned}$$

7. Since the matrix $\boldsymbol{A}$ is symmetric, the inverse of $\boldsymbol{A}$ is also symmetric. Then from the cofactor formula, we can have $\det \boldsymbol{A} = 4$ and

$$\begin{aligned} \left(\boldsymbol{A}^{-1}\right)_{11} &= \frac{\boldsymbol{C}_{11}}{\det \boldsymbol{A}} = \frac{\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}}{4} = \frac{3}{4} \\ \left(\boldsymbol{A}^{-1}\right)_{21} &= \frac{\boldsymbol{C}_{12}}{\det \boldsymbol{A}} = \frac{-\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix}}{4} = \frac{1}{2} \\ \left(\boldsymbol{A}^{-1}\right)_{22} &= \frac{\boldsymbol{C}_{22}}{\det \boldsymbol{A}} = \frac{\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}}{4} = 1 \\ \left(\boldsymbol{A}^{-1}\right)_{31} &= \frac{\boldsymbol{C}_{13}}{\det \boldsymbol{A}} = \frac{\begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix}}{4} = \frac{1}{4} \\ \left(\boldsymbol{A}^{-1}\right)_{32} &= \frac{\boldsymbol{C}_{23}}{\det \boldsymbol{A}} = \frac{-\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix}}{4} = \frac{1}{2} \\ \left(\boldsymbol{A}^{-1}\right)_{33} &= \frac{\boldsymbol{C}_{33}}{\det \boldsymbol{A}} = \frac{\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}}{4} = \frac{3}{4}. \end{aligned}$$

Therefore, we can obtain the inverse of $\boldsymbol{A}$ as

$$\boldsymbol{A}^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Similarly, since the matrix $\boldsymbol{B}$ is symmetric, the inverse of $\boldsymbol{B}$ is also symmetric. Then from the cofactor formula, we can have $\det \boldsymbol{B} = 1$ and

$$\left(\boldsymbol{B}^{-1}\right)_{11} = \frac{\boldsymbol{C}_{11}}{\det \boldsymbol{B}} = \frac{\begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix}}{1} = 2$$

$$\left(\boldsymbol{B}^{-1}\right)_{21} = \frac{\boldsymbol{C}_{12}}{\det \boldsymbol{B}} = \frac{-\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}}{1} = -1$$

$$\left(\boldsymbol{B}^{-1}\right)_{22} = \frac{\boldsymbol{C}_{22}}{\det \boldsymbol{B}} = \frac{\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}}{1} = 2$$

$$\left(\boldsymbol{B}^{-1}\right)_{31} = \frac{\boldsymbol{C}_{13}}{\det \boldsymbol{B}} = \frac{\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}}{1} = 0$$

$$\left(\boldsymbol{B}^{-1}\right)_{32} = \frac{\boldsymbol{C}_{23}}{\det \boldsymbol{B}} = \frac{-\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}}{1} = -1$$

$$\left(\boldsymbol{B}^{-1}\right)_{33} = \frac{\boldsymbol{C}_{33}}{\det \boldsymbol{B}} = \frac{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}}{1} = 1.$$

Therefore, we can obtain the inverse of $\boldsymbol{B}$ as

$$\boldsymbol{B}^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

8. For the first system, we have

$$\begin{bmatrix} 2 & 1 & -3 \\ 4 & 5 & 1 \\ -2 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 2 \end{bmatrix}.$$

Using Cramer's rule, we can obtain

$$x_1 = \frac{\begin{vmatrix} 0 & 1 & -3 \\ 8 & 5 & 1 \\ 2 & -1 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & -3 \\ 4 & 5 & 1 \\ -2 & -1 & 4 \end{vmatrix}} = 4,\ x_2 = \frac{\begin{vmatrix} 2 & 0 & -3 \\ 4 & 8 & 1 \\ -2 & 2 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & -3 \\ 4 & 5 & 1 \\ -2 & -1 & 4 \end{vmatrix}} = -2, \text{ and } x_3 = \frac{\begin{vmatrix} 2 & 1 & 0 \\ 4 & 5 & 8 \\ -2 & -1 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & -3 \\ 4 & 5 & 1 \\ -2 & -1 & 4 \end{vmatrix}} = 2.$$

For the second system, we have

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Using Cramer's rule, we can obtain

$$x_1 = \frac{\begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & -2 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \end{vmatrix}} = -\frac{2}{3},\ x_2 = \frac{\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \end{vmatrix}} = \frac{2}{3}$$

$$x_3 = \frac{\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \end{vmatrix}} = \frac{1}{3},\text{ and } x_4 = \frac{\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \end{vmatrix}} = 0.$$