EE 2030 Linear Algebra Chi-chao Chao
Spring 2011

Homework Assignment No. 5
Due 10:10am, June 1, 2011

`Reading:` Strang, Sections 6.1–6.5.

`Problems for Solution:`

1. (a) Suppose that $\boldsymbol{A}$ is an n by n matrix. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of $\boldsymbol{A}$, show that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \text{trace}(\boldsymbol{A}) = a_{11} + a_{22} + \cdots a_{nn}$$

where a_{ij} is the (i, j)th entry of $\boldsymbol{A}$, for $1 \leq i, j \leq n$.

   (b) A projection matrix $\boldsymbol{P}$ satisfies $\boldsymbol{P}^2 = \boldsymbol{P}$ and $\boldsymbol{P}^T = \boldsymbol{P}$. Show that the only possible eigenvalues of a projection matrix are 1 and 0.

2. Determine if each of the following matrices is diagonalizable. If it is, find an invertible matrix $\boldsymbol{S}$ and a diagonal matrix $\boldsymbol{\Lambda}$ such that $\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = \boldsymbol{\Lambda}$.

   (a) $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   (b) $\boldsymbol{A} = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$.

3. Suppose that

$$G_{k+2} = \frac{2}{3}G_{k+1} + \frac{1}{3}G_k, \quad \text{for } k \geq 0$$

with $G_0 = 0$ and $G_1 = 1$.

   (a) Find a general formula for G_k, $k \geq 0$.

   (b) Find $\lim_{k\to\infty} G_k$.

4. Suppose the rabbit population r and the wolf population w are governed by

$$\begin{aligned} \frac{dr}{dt} &= 4r - 2w \\ \frac{dw}{dt} &= r + w. \end{aligned}$$

   (a) Is this system stable, neutral, or unstable?

   (b) If initially $r = 300$ and $w = 200$, what are the populations of rabbits and wolves at time t?

(c) After a long time, what is the ratio of the rabbit population to the wolf population?

5. Consider

$$\boldsymbol{A} = \begin{bmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{bmatrix}.$$

(a) Find an orthogonal matrix $\boldsymbol{Q}$ and a diagonal matrix $\boldsymbol{\Lambda}$ such that $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T$.

(b) Find a_1, a_2 and $\boldsymbol{P}_1, \boldsymbol{P}_2$ such that

$$\boldsymbol{A} = a_1\boldsymbol{P}_1 + a_2\boldsymbol{P}_2$$

where $\boldsymbol{P}_1, \boldsymbol{P}_2$ are projection matrices.

6. Which of these classes of matrices do $\boldsymbol{A}$ and $\boldsymbol{B}$ belong to: Invertible, orthogonal, projection, permutation, symmetric, diagonalizable?

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

7. Determine if each of the following matrices is positive definite:

$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}^2.$$

8. The *Cholesky decomposition* says that if $\boldsymbol{A}$ is a positive definite matrix, then $\boldsymbol{A}$ can be factored into

$$\boldsymbol{A} = \boldsymbol{C}\boldsymbol{C}^T$$

where $\boldsymbol{C}$ is lower triangular with positive diagonal entries. Find the Cholesky decomposition for:

(a) $\boldsymbol{A} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{bmatrix}.$

(b) $\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 7 \end{bmatrix}.$