Solution to Homework Assignment No. 5

1. (a) We know that

$$\begin{aligned}
&\det(\boldsymbol{A}-\lambda \boldsymbol{I})\\
=&\begin{vmatrix} a_{11}-\lambda & a_{12} & & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & & \dots & a_{2n} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ a_{n1} & a_{n2} & & \dots & a_{nn}-\lambda \end{vmatrix}\\
=&(\lambda_1-\lambda)(\lambda_2-\lambda)\dots(\lambda_n-\lambda).
\end{aligned}$$

The only term in the big formula for $\det(\boldsymbol{A}-\lambda\boldsymbol{I})$ which contains the λ^{n-1} term is $(a_{11}-\lambda)(a_{22}-\lambda)\dots(a_{nn}-\lambda)$. Hence, the coefficient of λ^{n-1} in $\det(\boldsymbol{A}-\lambda\boldsymbol{I})$ is

$$(-1)^{n-1}(a_{11}+a_{22}+\dots+a_{nn})=(-1)^{n-1}\operatorname{trace}(\boldsymbol{A}).$$

On the other hand, the coefficient of λ^{n-1} in $(\lambda_1-\lambda)(\lambda_2-\lambda)\dots(\lambda_n-\lambda)$ is

$$(-1)^{n-1}(\lambda_1+\lambda_2+\dots+\lambda_n).$$

Therefore,

$$\lambda_1+\lambda_2+\dots+\lambda_n=\operatorname{trace}(\boldsymbol{A}).$$

(b) Assume that $\boldsymbol{P}$ has an eigenvalue λ and a corresponding eigenvector $\boldsymbol{x}$. We have $\boldsymbol{P}\boldsymbol{x}=\lambda\boldsymbol{x}$. With $\boldsymbol{P}^2=\boldsymbol{P}$, we can obtain $\boldsymbol{P}^2\boldsymbol{x}=\boldsymbol{P}\lambda\boldsymbol{x}=\lambda^2\boldsymbol{x}$ and therefore $\lambda\boldsymbol{x}=\lambda^2\boldsymbol{x}$. Since $\boldsymbol{x}$ is a nonzero vector, we have $\lambda^2=\lambda$, which implies that $\lambda=1$ or 0. Therefore, the only possible eigenvalues of a projection matrix are 1 and 0.

2. (a) A matrix $\boldsymbol{A}$ is diagonalizable if and only if each of its eigenvalues has the same algebraic multiplicity (AM) and geometric multiplicity (GM). Therefore, we have to find the eigenvalues and eigenvectors of $\boldsymbol{A}$.

$$\begin{aligned}
\det(\boldsymbol{A}-\lambda\boldsymbol{I})&=\begin{vmatrix} 1-\lambda & 0 & 9 \\ 0 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}\\
&=-(1-\lambda)^2\lambda=0.
\end{aligned}$$

Thus, we have $\lambda=1,1,0$. For $\lambda_1=1$, the AM of λ_1 equals 2. Besides,

$$\boldsymbol{A}-\lambda_1\boldsymbol{I}=\begin{bmatrix} 0 & 0 & 9 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the corresponding eigenvector is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

This gives that the GM of λ_1 equals 1 and is smaller than the AM of λ_1. As a result, $\boldsymbol{A}$ is not diagonalizable.

(b)

$$\det(\boldsymbol{A}-\lambda\boldsymbol{I}) = \begin{vmatrix} 5-\lambda & 0 & 0 \\ 1 & 5-\lambda & 0 \\ 0 & 1 & 5-\lambda \end{vmatrix} = (5-\lambda)^3 = 0.$$

Thus, we have $\lambda = 5, 5, 5$. It can be seen that the AM of λ equals 3. Besides,

$$\boldsymbol{A}-\lambda\boldsymbol{I} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and the corresponding eigenvector is

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This gives that the GM of λ equals 1 and is smaller than the AM of λ. As a result, $\boldsymbol{A}$ is not diagonalizable.

3. (a) Let $\boldsymbol{u}_k = \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$. The relation between $\boldsymbol{u}_{k+1} = \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix}$ and $\boldsymbol{u}_k = \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$ is given by

$$\boldsymbol{u}_{k+1} = \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}G_{k+1} + \frac{1}{3}G_k \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix} = \boldsymbol{A}\boldsymbol{u}_k.$$

Then we have $\boldsymbol{u}_k = \boldsymbol{A}\boldsymbol{u}_{k-1} = \boldsymbol{A}\boldsymbol{A}\boldsymbol{u}_{k-2} = \boldsymbol{A}^2\boldsymbol{u}_{k-2} = \boldsymbol{A}^k\boldsymbol{u}_0$. To find $\boldsymbol{A}^k$, we first find the eigenvalues of $\boldsymbol{A}$.

$$\begin{aligned} \det(\boldsymbol{A}-\lambda\boldsymbol{I}) &= \begin{vmatrix} \frac{2}{3}-\lambda & \frac{1}{3} \\ 1 & -\lambda \end{vmatrix} \\ &= \lambda^2 - \frac{2}{3}\lambda - \frac{1}{3} \\ &= (\lambda-1)\left(\lambda+\frac{1}{3}\right) = 0 \end{aligned}$$

$$\Longrightarrow \lambda = 1, -1/3.$$

For $\lambda_1 = 1$,

$$\boldsymbol{A}-\lambda_1\boldsymbol{I} = \begin{bmatrix} -1/3 & 1/3 \\ 1 & -1 \end{bmatrix}$$

and the corresponding eigenvector is

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = -1/3$,

$$\boldsymbol{A} - \lambda_2 \boldsymbol{I} = \begin{bmatrix} 1 & 1/3 \\ 1 & 1/3 \end{bmatrix}$$

and the corresponding eigenvector is

$$\boldsymbol{x}_2 = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}.$$

Therefore, we have

$$\boldsymbol{A} = \boldsymbol{S}\Lambda\boldsymbol{S}^{-1} = \begin{bmatrix} 1 & -1/3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & -1/3 \\ 1 & 1 \end{bmatrix}^{-1}.$$

We can write $\boldsymbol{u}_0$ as a linear combination of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ as follows:

$$\boldsymbol{u}_0 = \begin{bmatrix} G_1 \\ G_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1/3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3/4 \\ -3/4 \end{bmatrix}$$

$$\Longrightarrow \boldsymbol{u}_0 = \frac{3}{4}\boldsymbol{x}_1 - \frac{3}{4}\boldsymbol{x}_2.$$

Then we can obtain

$$\begin{aligned} \boldsymbol{u}_k &= \boldsymbol{A}^k \boldsymbol{u}_0 \\ &= \boldsymbol{A}^k \left(\frac{3}{4}\boldsymbol{x}_1 - \frac{3}{4}\boldsymbol{x}_2 \right) \\ &= \frac{3}{4} \left(1^k \boldsymbol{x}_1 - \left(-\frac{1}{3} \right)^k \boldsymbol{x}_2 \right) \\ &= \frac{3}{4} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \left(\frac{-1}{3} \right)^k \begin{bmatrix} -1/3 \\ 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}. \end{aligned}$$

Therefore, we can have

$$G_k = \frac{3}{4} - \frac{3}{4}\left(-\frac{1}{3} \right)^k$$

for $k \geq 0$.

(b) When k goes to infinity, the term $(-1/3)^k$ goes to zero. Therefore, we can obtain

$$\lim_{k \to \infty} G_k = \lim_{k \to \infty} \left(\frac{3}{4} - \frac{3}{4}\left(-\frac{1}{3} \right)^k \right) = \frac{3}{4}.$$

4. (a) Let $\boldsymbol{u} = \begin{bmatrix} r & w \end{bmatrix}^T$. We then have

$$\begin{aligned} \frac{d\boldsymbol{u}}{dt} &= \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \boldsymbol{u} \\ &= \boldsymbol{A}\boldsymbol{u}. \end{aligned}$$

To find the eigenvalues of $\boldsymbol{A}$, we calculate

$$\begin{aligned} \det(\boldsymbol{A} - \lambda \boldsymbol{I}) &= \begin{vmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} \\ &= \lambda^2 - 5\lambda + 6 \\ &= (\lambda - 2)(\lambda - 3). \end{aligned}$$

Therefore, $\lambda = 2, 3$. For $\lambda_1 = 2$,

$$\boldsymbol{A} - \lambda_1 \boldsymbol{I} = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$$

and the corresponding eigenvector is

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = 3$,

$$\boldsymbol{A} - \lambda_2 \boldsymbol{I} = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}$$

and the corresponding eigenvector is

$$\boldsymbol{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Hence, we have

$$\begin{aligned} \boldsymbol{u} &= \alpha e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \alpha e^{2t} + 2\beta e^{3t} \\ \alpha e^{2t} + \beta e^{3t} \end{bmatrix}. \end{aligned}$$

Since both the eigenvalues are positive, the system is unstable.

(b) At $t = 0$, we have

$$\begin{cases} \alpha + 2\beta = 300 \\ \alpha + \beta = 200. \end{cases}$$

Solving the equations gives $\alpha = \beta = 100$. Therefore, we have

$$r = 100e^{2t} + 200e^{3t}$$

and

$$w = 100e^{2t} + 100e^{3t}.$$

(c) After a long time, the ratio of the rabbit population to the wolf population can be obtained as

$$\begin{aligned}\lim_{t\to\infty}\frac{r}{w} &= \lim_{t\to\infty}\frac{100e^{2t}+200e^{3t}}{100e^{2t}+100e^{3t}} \\ &= 2.\end{aligned}$$

5. (a) To find an orthogonal matrix $\boldsymbol{Q}$, we first find the eigenvalues of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{bmatrix}.$$

We can have

$$\begin{aligned}\det(\boldsymbol{A}-\lambda\boldsymbol{I}) &= \begin{vmatrix} -\lambda & 2 & -1 \\ 2 & 3-\lambda & -2 \\ -1 & -2 & -\lambda \end{vmatrix} \\ &= \lambda^2(3-\lambda)+4+4+(\lambda-3)+4\lambda+4\lambda \\ &= -\lambda^3+3\lambda^2+9\lambda+5 \\ &= -(\lambda-5)(\lambda+1)^2 = 0.\end{aligned}$$

Therefore, we obtain $\lambda = 5, -1, -1$. For $\lambda_1 = 5$, we have

$$\boldsymbol{A}-\lambda_1\boldsymbol{I} = \begin{bmatrix} -5 & 2 & -1 \\ 2 & -2 & -2 \\ -1 & -2 & -5 \end{bmatrix}$$

and the corresponding unit eigenvector

$$\boldsymbol{x}_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = -1$, we have

$$\boldsymbol{A}-\lambda_2\boldsymbol{I} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix}.$$

and the corresponding eigenvectors

$$\boldsymbol{x}_2' = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\boldsymbol{x}_3' = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

Since $\boldsymbol{x}_2'$ and $\boldsymbol{x}_3'$ are not orthogonal, we use the Gram-Schmidt process to find orthonormal vectors $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ with the same span. We can obtain

$$\boldsymbol{x}_2 = \frac{\boldsymbol{x}_2'}{\|\boldsymbol{x}_2'\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}$$

and

$$\boldsymbol{x}_3 = \frac{\boldsymbol{x}_3' - (\boldsymbol{x}_2^T\boldsymbol{x}_3')\boldsymbol{x}_2}{\|\boldsymbol{x}_3' - (\boldsymbol{x}_2^T\boldsymbol{x}_3')\boldsymbol{x}_2\|} = \frac{1}{\sqrt{3}}\begin{bmatrix}-1\\1\\1\end{bmatrix}.$$

Therefore, we can obtain an orthogonal matrix

$$\boldsymbol{Q} = \frac{1}{\sqrt{6}}\begin{bmatrix}-1 & \sqrt{3} & -\sqrt{2}\\ -2 & 0 & \sqrt{2}\\ 1 & \sqrt{3} & \sqrt{2}\end{bmatrix}$$

and a diagonal matrix

$$\boldsymbol{\Lambda} = \begin{bmatrix}5 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}$$

such that $\boldsymbol{A} = \boldsymbol{Q\Lambda Q}^T$.

(b) As shown in (a), we can have

$$\begin{aligned}\boldsymbol{A} &= \boldsymbol{Q\Lambda Q}^T\\ &= \begin{bmatrix}\boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3\end{bmatrix}\begin{bmatrix}5 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_1^T\\ \boldsymbol{x}_2^T\\ \boldsymbol{x}_3^T\end{bmatrix}\\ &= 5(\boldsymbol{x}_1\boldsymbol{x}_1^T) + (-1)(\boldsymbol{x}_2\boldsymbol{x}_2^T + \boldsymbol{x}_3\boldsymbol{x}_3^T).\end{aligned}$$

Therefore, $a_1 = 5$, $a_2 = -1$,

$$\boldsymbol{P}_1 = \boldsymbol{x}_1\boldsymbol{x}_1^T = \begin{bmatrix}1/6 & 1/3 & -1/6\\ 1/3 & 2/3 & -1/3\\ -1/6 & -1/3 & 1/6\end{bmatrix}$$

and

$$\boldsymbol{P}_2 = \boldsymbol{x}_2\boldsymbol{x}_2^T + \boldsymbol{x}_3\boldsymbol{x}_3^T = \begin{bmatrix}5/6 & -1/3 & 1/6\\ -1/3 & 1/3 & 1/3\\ 1/6 & 1/3 & 5/6\end{bmatrix}.$$

6. (a) For $\boldsymbol{A}$, it can be seen that $\dim(\mathcal{C}(\boldsymbol{A})) = 4$ where $\dim(\mathcal{C}(\boldsymbol{A}))$ is the dimension of the column space of $\boldsymbol{A}$, so $\boldsymbol{A}$ is invertible. All the column vectors of $\boldsymbol{A}$ are of unit length and mutually orthogonal, so $\boldsymbol{A}$ is orthogonal. Since $\boldsymbol{A} \neq \boldsymbol{A}^T$, $\boldsymbol{A}$ is not a projection matrix. The rows of $\boldsymbol{A}$ are a permutation of those of the identity matrix, so $\boldsymbol{A}$ is a permutation matrix. Since $\boldsymbol{A} \neq \boldsymbol{A}^T$ as mentioned before, $\boldsymbol{A}$ is not symmetric. Besides,

$$
\begin{aligned}
\det(\boldsymbol{A}-\lambda\boldsymbol{I}) &= \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} \\
&= (-1)\begin{vmatrix} 1 & 0 & 0 \\ -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \end{vmatrix} + (-\lambda)\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} \\
&= -1+\lambda^4 = 0
\end{aligned}
$$

gives that $\lambda = \pm 1, \pm i$. Since $\boldsymbol{A}$ has four different eigenvalues, $\boldsymbol{A}$ is diagonalizable.

(b) For $\boldsymbol{B}$, it can be seen that $\dim(\mathcal{C}(\boldsymbol{B})) = 1$, so $\boldsymbol{B}$ is not invertible. The first column of $\boldsymbol{B}$ is not orthogonal to the second column of $\boldsymbol{B}$, so $\boldsymbol{B}$ is not orthogonal. Since $B = \boldsymbol{x}\boldsymbol{x}^T$, where $\boldsymbol{x} = \frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$, $\boldsymbol{B}$ is a projection matrix. The rows of $\boldsymbol{B}$ are not a permutation of those of the identity matrix, so $\boldsymbol{B}$ is not a permutation matrix. Since $\boldsymbol{B} = \boldsymbol{B}^T$, $\boldsymbol{B}$ is symmetric. Besides, since $\boldsymbol{B}$ is symmetric, it is diagonalizable by the Spectral Theorem.

7. To determine whether a matrix is positive definite, we can check whether all the upper left determinants are positive. For $\boldsymbol{A}$, we have $\det(\boldsymbol{A}) = 0$, so $\boldsymbol{A}$ is not positive definite. For $\boldsymbol{B}$, $2 > 0$,

$$
\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0
$$

and

$$
\begin{vmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{vmatrix} = 4 > 0.
$$

Therefore, $\boldsymbol{B}$ is positive definite. It can be verified that

$$
\boldsymbol{C} = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}.
$$

Similarly, it can be checked that $\boldsymbol{C}$ is positive definite by verifying that all the upper left determinants are positive.

8. (a) By Gaussian elimination, we can obtain

$$
\begin{aligned}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \boldsymbol{A} &= \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\boldsymbol{A} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T.
\end{aligned}
$$

We then have $\boldsymbol{A} = \boldsymbol{C}\boldsymbol{C}^T$ where

$$
\begin{aligned}
\boldsymbol{C} &= \boldsymbol{L}\sqrt{\boldsymbol{D}} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}.
\end{aligned}
$$

(b) Similar to (a), by Gaussian elimination, we can obtain

$$
\begin{aligned}
\boldsymbol{A} &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T.
\end{aligned}
$$

We then have $\boldsymbol{A} = \boldsymbol{C}\boldsymbol{C}^T$ where

$$
\begin{aligned}
\boldsymbol{C} &= \boldsymbol{L}\sqrt{\boldsymbol{D}} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & \sqrt{5} \end{bmatrix}.
\end{aligned}
$$