EE 2030 Linear Algebra Chi-chao Chao
Spring 2011

Homework Assignment No. 6
Due 10:10am, June 15, 2011

`Reading:` Strang, Sections 6.6, 6.7, 7.1, 7.2, 7.3 (up to the top half of p. 401).

`Problems for Solution:`

1. Determine if each of the following statements is true. If yes, prove it. Otherwise, show why it is not or find a counterexample.

   (a) If $\boldsymbol{A}$ is similar to $\boldsymbol{B}$, then $\boldsymbol{A}^2$ is similar to $\boldsymbol{B}^2$.

   (b) If $\boldsymbol{A}^2$ is similar to $\boldsymbol{B}^2$, then $\boldsymbol{A}$ is similar to $\boldsymbol{B}$.

   (c) $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ is similar to $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$.

   (d) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ is similar to $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$.

2. For 4 by 4 matrices with eigenvalues $0, 0, 0, 0$, there are five different Jordan forms. Find all of them.

3. Consider

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

   (a) Compute $\boldsymbol{A}^T\boldsymbol{A}$ and $\boldsymbol{A}\boldsymbol{A}^T$ and their eigenvalues and unit eigenvectors.

   (b) Construct the singular value decomposition (SVD) and verify that $\boldsymbol{A}$ equals $\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$.

   (c) Find orthonormal bases for the four fundamental subspaces of $\boldsymbol{A}$.

4. Suppose $\boldsymbol{A}$ has orthogonal columns $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n$ of lengths $\sigma_1, \sigma_2, \ldots, \sigma_n$. What are $\boldsymbol{U}$, $\boldsymbol{\Sigma}$, and $\boldsymbol{V}$ in the SVD?

5. A linear transformation T from V to W has an *inverse* from W to V when the range is all of W and the kernel contains only $\boldsymbol{v} = \boldsymbol{0}$. Then $T(\boldsymbol{v}) = \boldsymbol{w}$ has one solution $\boldsymbol{v}$ for each $\boldsymbol{w}$ in W. Why are these T's not invertible?

   (a) $T(v_1, v_2) = (v_2, v_2)$, where $W = \mathcal{R}^2$.

   (b) $T(v_1, v_2) = (v_1, v_2, v_1 + v_2)$, where $W = \mathcal{R}^3$.

   (c) $T(v_1, v_2) = v_1$, where $W = \mathcal{R}^1$.

6. Consider the vector space V spanned by the basis functions $1, x, x^2, x^3$. The linear operator S on V takes the *second derivative*. Find the 4 by 4 matrix representation $\boldsymbol{B}$ for S with respect to this basis.

7. Suppose $\boldsymbol{A}$ is a 3 by 4 matrix of rank $r = 2$, and the linear transformation $T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v}$. Choose input basis vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ from the row space of $\boldsymbol{A}$ and $\boldsymbol{v}_3, \boldsymbol{v}_4$ from the nullspace of $\boldsymbol{A}$. Choose output basis vectors $\boldsymbol{w}_1 = \boldsymbol{A}\boldsymbol{v}_1, \boldsymbol{w}_2 = \boldsymbol{A}\boldsymbol{v}_2$ in the column space of $\boldsymbol{A}$ and $\boldsymbol{w}_3$ from the left nullspace of $\boldsymbol{A}$. What matrix represents this T in these special bases?

8. Define the linear operator T on $\mathcal{R}^2$ by

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 + v_2 \\ v_1 + v_2 \end{bmatrix}.$$

Find a basis for $\mathcal{R}^2$ such that the matrix representation for T in this basis is a diagonal matrix.