Solution to Homework Assignment No. 6

1. (a) It is true. Suppose $\boldsymbol{A}$ is similar to $\boldsymbol{B}$. We then have $\boldsymbol{A} = \boldsymbol{M}^{-1}\boldsymbol{B}\boldsymbol{M}$ for some matrix $\boldsymbol{M}$ and hence $\boldsymbol{A}^2 = \boldsymbol{M}^{-1}\boldsymbol{B}^2\boldsymbol{M}$. Therefore, $\boldsymbol{A}^2$ is similar to $\boldsymbol{B}^2$.

(b) It is not true. Consider $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Since $\boldsymbol{A}^2 = \boldsymbol{B}^2$, $\boldsymbol{A}^2$ is similar to $\boldsymbol{B}^2$. However, $\boldsymbol{A}$ is not similar to $\boldsymbol{B}$.

(c) It is true. Since the eigenvalues of $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$ are 3 and 4, $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$ can be diagonalized to $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$. Hence $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ is similar to $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$.

(d) It is not true. A simple check reveals that $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalizable. Therefore, $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ cannot be similar.

2. Considering the geometry multiplicity (GM) of the eigenvalue 0, we have four different cases:

$$\text{GM} = 1 \quad \Longrightarrow \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\text{GM} = 2 \quad \Longrightarrow \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\text{GM} = 3 \quad \Longrightarrow \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\text{GM} = 4 \quad \Longrightarrow \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

These are the five different Jordan forms.

3. (a) After some calculations, we can obtain the eigenvalues and unit eigenvectors of $\boldsymbol{A}^T\boldsymbol{A}$ and $\boldsymbol{A}\boldsymbol{A}^T$ as follows:

$$\boldsymbol{A}^T\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \implies \begin{cases} \lambda_1 = 3 \longleftrightarrow \boldsymbol{v}_1 = \frac{1}{\sqrt{6}}(1,2,1)^T \\ \lambda_2 = 1 \longleftrightarrow \boldsymbol{v}_2 = \frac{1}{\sqrt{2}}(1,0,-1)^T \\ \lambda_3 = 0 \longleftrightarrow \boldsymbol{v}_3 = \frac{1}{\sqrt{3}}(1,-1,1)^T. \end{cases}$$

$$\boldsymbol{A}\boldsymbol{A}^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \implies \begin{cases} \lambda_1 = 3 \longleftrightarrow \boldsymbol{u}_1 = \frac{1}{\sqrt{2}}(1,1)^T \\ \lambda_2 = 1 \longleftrightarrow \boldsymbol{u}_2 = \frac{1}{\sqrt{2}}(1,-1)^T. \end{cases}$$

(b) According to (a), the singular value decomposition of $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$$

where

$$\boldsymbol{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{V} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & \sqrt{3} & \sqrt{2} \\ 2 & 0 & -\sqrt{2} \\ 1 & -\sqrt{3} & \sqrt{2} \end{bmatrix}.$$

The decomposition can be verified by

$$\begin{aligned} \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & \sqrt{3} & \sqrt{2} \\ 2 & 0 & -\sqrt{2} \\ 1 & -\sqrt{3} & \sqrt{2} \end{bmatrix}^T \\ &= \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{3} & 1 & 0 \\ \sqrt{3} & -1 & 0 \end{bmatrix} \cdot \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & 2 & 1 \\ \sqrt{3} & 0 & -\sqrt{3} \\ \sqrt{2} & -\sqrt{2} & \sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \boldsymbol{A}. \end{aligned}$$

(c) According to what was taught in class, we know that we can choose the unit eigenvectors obtained in (a) to form orthonormal bases for the four fundamental subspaces of $\boldsymbol{A}$. Therefore, $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, $\{\boldsymbol{v}_3\}$, $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, and ϕ are orthonormal bases for $\mathcal{C}(\boldsymbol{A}^T)$, $\mathcal{N}(\boldsymbol{A})$, $\mathcal{C}(\boldsymbol{A})$, and $\mathcal{N}(\boldsymbol{A}^T)$, respectively. Note that the basis for the zero space is the empty set.

4. Since $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_n \end{bmatrix}$ has orthogonal columns, we have

$$\boldsymbol{A}^T\boldsymbol{A} = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n^2 \end{bmatrix} \implies \begin{cases} \lambda_1 = \sigma_1^2 \longleftrightarrow \boldsymbol{v}_1 = (1,0,\cdots,0)^T \\ \lambda_2 = \sigma_2^2 \longleftrightarrow \boldsymbol{v}_2 = (0,1,\cdots,0)^T \\ \quad \vdots \qquad\quad \vdots \qquad\qquad \vdots \\ \lambda_n = \sigma_n^2 \longleftrightarrow \boldsymbol{v}_n = (0,0,\cdots,1)^T. \end{cases}$$

On the other hand, we have $\boldsymbol{u}_i = \boldsymbol{A}\boldsymbol{v}_i/\sigma_i = \boldsymbol{w}_i/\sigma_i$ for $i = 1, 2, \cdots, n$. Therefore, we can obtain in the SVD

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{w}_1/\sigma_1 & \boldsymbol{w}_2/\sigma_2 & \cdots & \boldsymbol{w}_n/\sigma_n \end{bmatrix},$$

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix}, \quad \boldsymbol{V} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

5. (a) Since $T(1,0) = (0,0)$, this T is not invertible.

(b) Since $(1,0,0)$ is not in the range, this T is not invertible.

(c) Since $T(0,1) = 0$, this T is not invertible.

6. For the second derivative, we have $S(1) = 0$, $S(x) = 0$, $S(x^2) = 2$, and $S(x^3) = 6x$. Then we can obtain

$$\boldsymbol{B} = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

7. We know that $T(\boldsymbol{v}_1) = \boldsymbol{A}\boldsymbol{v}_1 = \boldsymbol{w}_1$, $T(\boldsymbol{v}_2) = \boldsymbol{A}\boldsymbol{v}_2 = \boldsymbol{w}_2$, $T(\boldsymbol{v}_3) = \boldsymbol{A}\boldsymbol{v}_3 = \mathbf{0}$, and $T(\boldsymbol{v}_4) = \boldsymbol{A}\boldsymbol{v}_4 = \mathbf{0}$. Therefore, the matrix which represents this T is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

8. For the standard basis, the matrix which represents this T is

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

The eigenvectors for this matrix are $(1,1)$ and $(1,-1)$. Therefore, we can find the basis $\{(1,1),(1,-1)\}$ such that the matrix representation for T in this basis is a diagonal matrix.