EE 2030 Linear Algebra Chi-chao Chao
Spring 2012

Homework Assignment No. 1
Due 10:10am, March 9, 2012

`Reading:` Strang, Chapters 1 and 2.

`Problems for Solution:`

1. Find the pivots and solutions for both systems of linear equations:

$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 5 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 20 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & -3 & 0 \\ 4 & -5 & 1 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}.$$

2. Find $\boldsymbol{A}^{-1}$ and $\boldsymbol{B}^{-1}$ if they exist:

$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$

3. True or false. (If it is true, prove it. Otherwise, find a counterexample.)
   (a) Let $\boldsymbol{A}$ be an invertible matrix. If a matrix $\boldsymbol{B}$ satisfies $\boldsymbol{AB} = \boldsymbol{I}$, then $\boldsymbol{B} = \boldsymbol{A}^{-1}$.
   (b) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are both invertible, then $\boldsymbol{A} + \boldsymbol{B}$ is invertible.
   (c) Assume $\boldsymbol{A}$ is invertible. If $\boldsymbol{A}$ is not symmetric, then $\boldsymbol{A}^{-1}$ is not symmetric.

4. Factor the following matrices into $\boldsymbol{A} = \boldsymbol{LDU}$:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 5 & 6 \end{bmatrix} \quad \text{and} \quad \boldsymbol{A} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

5. (a) Under what conditions is $\boldsymbol{A}$ nonsingular, if $\boldsymbol{A}$ is the product

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}?$$

   (b) Solve

$$\boldsymbol{LUx} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

   without multiplying $\boldsymbol{LU}$ to find $\boldsymbol{A}$.

6. If $\boldsymbol{A} = \boldsymbol{L}_1\boldsymbol{D}_1\boldsymbol{U}_1$ and $\boldsymbol{A} = \boldsymbol{L}_2\boldsymbol{D}_2\boldsymbol{U}_2$, where the $\boldsymbol{L}$'s are lower triangular with unit diagonal, the $\boldsymbol{U}$'s are upper triangular with unit diagonal, and $\boldsymbol{D}$'s are diagonal matrices with no zeros on the diagonal, prove that $\boldsymbol{L}_1 = \boldsymbol{L}_2$, $\boldsymbol{D}_1 = \boldsymbol{D}_2$, and $\boldsymbol{U}_1 = \boldsymbol{U}_2$. Note that the proof can be decomposed into the following two steps:

   (a) Derive the equation $\boldsymbol{L}_1^{-1}\boldsymbol{L}_2\boldsymbol{D}_2 = \boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1}$ and explain why one side is lower triangular and the other side is upper triangular.

   (b) Compare the main diagonals in the equation in (a), and then compare the off-diagonals.

   In your proof, you may use the following assertions without proving them:

   (i) A lower (upper) triangular matrix with unit diagonal is invertible and its inverse is still lower (upper) triangular with unit diagonal.

   (ii) The product of two lower (upper) triangular matrices with unit diagonal is still lower (upper) triangular with unit diagonal.

   (iii) The product of a lower (upper) triangular matrix and a diagonal matrix is lower (upper) triangular.

7. (a) Given

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

   find 3 by 3 matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ such that

$$\boldsymbol{A} = \boldsymbol{B} + \boldsymbol{C}$$

   where $\boldsymbol{B}$ is symmetric and $\boldsymbol{C}$ is skew-symmetric. (Note that a matrix $\boldsymbol{C}$ is called *skew-symmetric* if $\boldsymbol{C}^T = -\boldsymbol{C}$.)

   (b) Now given a general n by n matrix $\boldsymbol{A}$, find n by n matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ such that

$$\boldsymbol{A} = \boldsymbol{B} + \boldsymbol{C}$$

   where $\boldsymbol{B}$ is symmetric and $\boldsymbol{C}$ is skew-symmetric. (*Hint*: Express $\boldsymbol{B}$ and $\boldsymbol{C}$ in terms of $\boldsymbol{A}$ and $\boldsymbol{A}^T$.)

8. Factor the following matrix into $\boldsymbol{PA} = \boldsymbol{LU}$. Also factor it into $\boldsymbol{A} = \boldsymbol{L}_1\boldsymbol{P}_1\boldsymbol{U}_1$.

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 6 \end{bmatrix}.$$