Solution to Homework Assignment No. 1

1. (a) We first perform forward elimination:

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 4 & 7 & 5 & 20 \\ 0 & -2 & 2 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 0 & 1 & 3 & 4 \\ 0 & -2 & 2 & 0 \end{array}\right] \text{(subtract } 2 \times \text{row 1)}$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 8 & 8 \end{array}\right] \text{(add } 2 \times \text{row 2).}$$

Then we obtain the pivots as 2, 1, and 8, and the solution can be solved by back substitution as follows:

$$\begin{array}{lll} \text{equation 3:} & 8z = 8 & \text{gives } z = 1 \\ \text{equation 2:} & 1y + 3 = 4 & \text{gives } y = 1 \\ \text{equation 1:} & 2x + 3 + 1 = 8 & \text{gives } x = 2. \end{array}$$

We have $(x, y, z) = (2, 1, 1)$.

(b) We perform forward elimination first:

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 4 & -5 & 1 & 7 \\ 2 & -1 & -3 & 5 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 2 & -1 & -3 & 5 \end{array}\right] \text{(subtract } 2 \times \text{row 1)}$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & -3 & 2 \end{array}\right] \text{(subtract } 1 \times \text{row 1)}$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & 0 \end{array}\right] \text{(subtract } 2 \times \text{row 2).}$$

The pivots are 2, 1, and -5. We then do back substitution to get the solution:

$$\begin{array}{lll} \text{equation 3:} & -5z = 0 & \text{gives } z = 0 \\ \text{equation 2:} & y + 0 = 1 & \text{gives } y = 1 \\ \text{equation 1:} & 2x - 3 = 3 & \text{gives } x = 3. \end{array}$$

The solution is $(x, y, z) = (3, 1, 0)$.

2. (a) We use the *Gauss-Jordan method* to find the inverse of $\boldsymbol{A}$:

$$
\left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{I} \end{array}\right] = \left[\begin{array}{cccc|cccc}
2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 & 0 & 1
\end{array}\right]
$$

$$
\overset{\boldsymbol{E}_{21}}{\Longrightarrow} \left[\begin{array}{cccc|cccc}
2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 3/2 & -1 & 0 & 1/2 & 1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 & 0 & 1
\end{array}\right]
$$

$$
\overset{\boldsymbol{E}_{32}}{\Longrightarrow} \left[\begin{array}{cccc|cccc}
2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 3/2 & -1 & 0 & 1/2 & 1 & 0 & 0 \\
0 & 0 & 4/3 & -1 & 1/3 & 2/3 & 1 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 & 0 & 1
\end{array}\right]
$$

$$
\overset{\boldsymbol{E}_{43}}{\Longrightarrow} \left[\begin{array}{cccc|cccc}
2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 3/2 & -1 & 0 & 1/2 & 1 & 0 & 0 \\
0 & 0 & 4/3 & -1 & 1/3 & 2/3 & 1 & 0 \\
0 & 0 & 0 & 5/4 & 1/4 & 1/2 & 3/4 & 1
\end{array}\right]
$$

$$
\overset{\boldsymbol{E}_{34}}{\Longrightarrow} \left[\begin{array}{cccc|cccc}
2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 3/2 & -1 & 0 & 1/2 & 1 & 0 & 0 \\
0 & 0 & 4/3 & 0 & 8/15 & 16/15 & 8/5 & 4/5 \\
0 & 0 & 0 & 5/4 & 1/4 & 1/2 & 3/4 & 1
\end{array}\right]
$$

$$
\overset{\boldsymbol{E}_{23}}{\Longrightarrow} \left[\begin{array}{cccc|cccc}
2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 3/2 & 0 & 0 & 9/10 & 9/5 & 6/5 & 3/5 \\
0 & 0 & 4/3 & 0 & 8/15 & 16/15 & 8/5 & 4/5 \\
0 & 0 & 0 & 5/4 & 1/4 & 1/2 & 3/4 & 1
\end{array}\right]
$$

$$
\overset{\boldsymbol{E}_{12}}{\Longrightarrow} \left[\begin{array}{cccc|cccc}
2 & 0 & 0 & 0 & 8/5 & 6/5 & 4/5 & 2/5 \\
0 & 3/2 & 0 & 0 & 9/10 & 9/5 & 6/5 & 3/5 \\
0 & 0 & 4/3 & 0 & 8/15 & 16/5 & 8/5 & 4/5 \\
0 & 0 & 0 & 5/4 & 1/4 & 1/2 & 3/4 & 1
\end{array}\right]
$$

$$
\overset{\boldsymbol{D}^{-1}}{\Longrightarrow} \left[\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & 4/5 & 3/5 & 2/5 & 1/5 \\
0 & 1 & 0 & 0 & 3/5 & 6/5 & 4/5 & 2/5 \\
0 & 0 & 1 & 0 & 2/5 & 4/5 & 6/5 & 3/5 \\
0 & 0 & 0 & 1 & 1/5 & 2/5 & 3/5 & 4/5
\end{array}\right]
$$

where

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2/3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3/4 & 1 \end{bmatrix}$$

$$E_{34} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4/5 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_{23} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3/4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 1 & 2/3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 0 \\ 0 & 0 & 3/4 & 0 \\ 0 & 0 & 0 & 4/5 \end{bmatrix}.$$

Thus we have

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

(b) We use the *Gauss-Jordan method* to find B^{-1}:

$$[\, A \mid I \,] = \left[\begin{array}{cccc|cccc} 2 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\Longrightarrow}{E_{21}} \left[\begin{array}{cccc|cccc} 2 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & -1/2 & 1/2 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\Longrightarrow}{E_{41}} \left[\begin{array}{cccc|cccc} 2 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & -1/2 & 1/2 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1/2 & -1 & 3/2 & 1/2 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{\boldsymbol{E}_{32}}{\Longrightarrow}\left[\begin{array}{cccc|cccc} 2 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & -1/2 & 1/2 & 1 & 0 & 0 \\ 0 & 0 & 4/3 & -4/3 & 1/3 & 2/3 & 1 & 0 \\ 0 & -1/2 & -1 & 3/2 & 1/2 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{\boldsymbol{E}_{42}}{\Longrightarrow}\left[\begin{array}{cccc|cccc} 2 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & -1/2 & 1/2 & 1 & 0 & 0 \\ 0 & 0 & 4/3 & -4/3 & 1/3 & 2/3 & 1 & 0 \\ 0 & 0 & -4/3 & 4/3 & 2/3 & 1/3 & 0 & 1 \end{array}\right]$$

$$\overset{\boldsymbol{E}_{43}}{\Longrightarrow}\left[\begin{array}{cccc|cccc} 2 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & -1/2 & 1/2 & 1 & 0 & 0 \\ 0 & 0 & 4/3 & -4/3 & 1/3 & 2/3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right].$$

Since we can not find a full set of *nonzero* pivots, $\boldsymbol{B}$ is *not* invertible.

Alternatively, we can prove by contradiction that it has no inverse. Suppose $\boldsymbol{B}^{-1}$ exists. Let $\boldsymbol{a} = [1 \quad 1 \quad 1 \quad 1]$. We have

$$\boldsymbol{a}\left(\boldsymbol{B}\boldsymbol{B}^{-1}\right) = \boldsymbol{a}\boldsymbol{I} = \boldsymbol{a}$$
$$\left(\boldsymbol{a}\boldsymbol{B}\right)\boldsymbol{B}^{-1} = \boldsymbol{0}\boldsymbol{B}^{-1} = \boldsymbol{0}$$

where $\boldsymbol{0}$ is the 1×4 zero vector. Since the *associative law* is violated, we get a contradiction. Therefore, $\boldsymbol{B}^{-1}$ does *not* exist.

3. (a) True.
$\boldsymbol{B} = \boldsymbol{I}\boldsymbol{B} = (\boldsymbol{A}^{-1}\boldsymbol{A})\boldsymbol{B} = \boldsymbol{A}^{-1}(\boldsymbol{A}\boldsymbol{B}) = \boldsymbol{A}^{-1}\boldsymbol{I} = \boldsymbol{A}^{-1} \Rightarrow \boldsymbol{B} = \boldsymbol{A}^{-1}$.

(b) False.
Consider

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \boldsymbol{B} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Both matrices are invertible. But

$$\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which is not invertible.

(c) True.
This is equivalent to showing that $\boldsymbol{A}$ is symmetric if $\boldsymbol{A}^{-1}$ is symmetric. Suppose $\boldsymbol{A}^{-1}$ is symmetric. We have

$$\begin{aligned} &\boldsymbol{A}^{-1} = (\boldsymbol{A}^{-1})^T \\ \Rightarrow &\boldsymbol{I} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}(\boldsymbol{A}^{-1})^T \\ \Rightarrow &\boldsymbol{A}^T = \boldsymbol{I}\boldsymbol{A}^T = \boldsymbol{A}(\boldsymbol{A}^{-1})^T\boldsymbol{A}^T \\ &\quad = \boldsymbol{A}(\boldsymbol{A}\boldsymbol{A}^{-1})^T = \boldsymbol{A}\boldsymbol{I}^T = \boldsymbol{A}\boldsymbol{I} = \boldsymbol{A} \end{aligned}$$

which shows that $\boldsymbol{A}$ is symmetric.

4. (a)

$$\boldsymbol{E}_{21}\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 5 & 6 \end{bmatrix}.$$

$$\boldsymbol{E}_{31}\boldsymbol{E}_{21}\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 4 \end{bmatrix}.$$

$$\boldsymbol{U} = \boldsymbol{E}_{32}\boldsymbol{E}_{31}\boldsymbol{E}_{21}\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 5 \end{bmatrix}.$$

Thus we have

$$\boldsymbol{L} = \boldsymbol{E}_{21}^{-1}\boldsymbol{E}_{31}^{-1}\boldsymbol{E}_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad \boldsymbol{U} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) As we did in part (a) of Problem 2, we can find that

$$\boldsymbol{L} = \boldsymbol{E}_{21}^{-1}\boldsymbol{E}_{32}^{-1}\boldsymbol{E}_{43}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 4/3 & 0 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}$$

$$\boldsymbol{U} = \begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 1 & -3/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

5. (a) Observe that d_1, d_2, d_3 are pivots. In order to have a full set of pivots, we should have $d_1 d_2 d_3 \neq 0$.

(b) We first solve $\boldsymbol{Lc} = \boldsymbol{y}$:

$$\boldsymbol{Lc} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = \boldsymbol{c}.$$

Then we solve $\boldsymbol{Ux} = \boldsymbol{c}$:

$$\begin{array}{lcl} \text{equation 3:} & 1 \cdot w = 0 & \text{gives } w = 0 \\ \text{equation 2:} & 1 \cdot v + 0 = -2 & \text{gives } v = -2 \\ \text{equation 1:} & 2 \cdot u + 4 \cdot (-2) + 0 = 2 & \text{gives } u = 5. \end{array}$$

We have $(u, v, w) = (5, -2, 0)$.

6. (a) By (i) both $\boldsymbol{L}_1^{-1}$ and $\boldsymbol{U}_2^{-1}$ exist. We can have

$$\begin{aligned} \boldsymbol{L}_1\boldsymbol{D}_1\boldsymbol{U}_1 &= \boldsymbol{L}_2\boldsymbol{D}_2\boldsymbol{U}_2 \\ \Longrightarrow \boldsymbol{D}_1\boldsymbol{U}_1 &= \boldsymbol{L}_1^{-1}\boldsymbol{L}_2\boldsymbol{D}_2\boldsymbol{U}_2 \\ \Longrightarrow \boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1} &= \boldsymbol{L}_1^{-1}\boldsymbol{L}_2\boldsymbol{D}_2. \end{aligned}$$

By (i) and (ii) we have that $\boldsymbol{L}_1^{-1}\boldsymbol{L}_2$ a lower triangular matrix with unit diagonal. Also by (iii) $\boldsymbol{L}_1^{-1}\boldsymbol{L}_2\boldsymbol{D}_2$ is a lower triangular matrix. Similarly, $\boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1}$ is upper triangular.

(b) Observing the left hand side, we have $\boldsymbol{U}_1\boldsymbol{U}_2^{-1}$ is an upper triangular matrix with unit diagonal; i.e. $(\boldsymbol{U}_1\boldsymbol{U}_2^{-1})_{ii} = 1$, for all i. Now we consider the diagonal terms of $\boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1}$. We find that $(\boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1})_{ii} = \sum_j (\boldsymbol{D}_1)_{ij}(\boldsymbol{U}_1\boldsymbol{U}_2^{-1})_{ji} = (\boldsymbol{D}_1)_{ii}(\boldsymbol{U}_1\boldsymbol{U}_2^{-1})_{ii}$, since $(\boldsymbol{D}_1)_{ij} = 0$, $\forall i \neq j$. From the above we can deduce that $(\boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1})_{ii} = (\boldsymbol{D}_1)_{ii}$. Similarly, $(\boldsymbol{L}_1^{-1}\boldsymbol{L}_2\boldsymbol{D}_2)_{ii} = (\boldsymbol{D}_2)_{ii}$. We can then obtain the fact that $(\boldsymbol{D}_1)_{ii} = (\boldsymbol{D}_2)_{ii}$, for all i. Therefore, $\boldsymbol{D}_1 = \boldsymbol{D}_2$.

Now comes the off-diagonals. From part (a) we have an lower triangular matrix equal to an upper triangular matrix. The only possibility is that $\boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1} = \boldsymbol{L}_1^{-1}\boldsymbol{L}_2\boldsymbol{D}_2$ is a diagonal matrix, which means both $\boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1}$ and $\boldsymbol{L}_1^{-1}\boldsymbol{L}_2\boldsymbol{D}_2$ only have non-zero values on the main diagonal. From the previous paragraph we have learned that the values on the main diagonal of the above two matrices are the same as those of $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$. Thus we have $\boldsymbol{D}_1 = \boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1} = \boldsymbol{D}_2 = \boldsymbol{L}_1^{-1}\boldsymbol{L}_2\boldsymbol{D}_2$. Since $\boldsymbol{D}_1$, $\boldsymbol{D}_2$ are invertible, we can then have $\boldsymbol{U}_1\boldsymbol{U}_2^{-1} = \boldsymbol{L}_1^{-1}\boldsymbol{L}_2 = \boldsymbol{I}$, which gives $\boldsymbol{L}_1 = \boldsymbol{L}_2$ and $\boldsymbol{U}_1 = \boldsymbol{U}_2$, because of the fact that the inverses are unique.

7. (a) We can have

$$\boldsymbol{A}+\boldsymbol{A^T}=\boldsymbol{B}+\boldsymbol{B^T}+\boldsymbol{C}+\boldsymbol{C}^T=\boldsymbol{B}+\boldsymbol{B}+\boldsymbol{C}-\boldsymbol{C}=\boldsymbol{2B}$$

$$\Rightarrow \boldsymbol{2B}=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}+\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}=\begin{bmatrix}2&6&10\\6&10&14\\10&14&18\end{bmatrix}$$

$$\Rightarrow \boldsymbol{B}=\begin{bmatrix}1&3&5\\3&5&7\\5&7&9\end{bmatrix}$$

$$\Rightarrow \boldsymbol{C}=\boldsymbol{A}-\boldsymbol{B}=\begin{bmatrix}0&-1&-2\\1&0&-1\\2&1&0\end{bmatrix}.$$

(b) We can generalize the method in part (a) to obtain $\boldsymbol{B}=(\boldsymbol{A}+\boldsymbol{A}^T)/2$ and $\boldsymbol{C}=(\boldsymbol{A}-\boldsymbol{A}^T)/2$.

8. (a) We first perform row exchange to obtain

$$\boldsymbol{PA}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&1\\1&0&1\\2&3&6\end{bmatrix}=\begin{bmatrix}1&0&1\\0&1&1\\2&3&6\end{bmatrix}=\boldsymbol{A}'.$$

Then elimination gives

$$\boldsymbol{E}_{32}\boldsymbol{E}_{31}\boldsymbol{A}'=\begin{bmatrix}1&0&0\\0&1&0\\0&-3&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}\begin{bmatrix}1&0&1\\0&1&1\\2&3&6\end{bmatrix}=\begin{bmatrix}1&0&1\\0&1&1\\0&0&1\end{bmatrix}=\boldsymbol{U}.$$

Therefore, we can obtain $\boldsymbol{PA}=\boldsymbol{LU}$ where

$$\boldsymbol{P}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},\quad \boldsymbol{L}=\boldsymbol{E}_{31}^{-1}\boldsymbol{E}_{32}^{-1}=\begin{bmatrix}1&0&0\\0&1&0\\2&3&1\end{bmatrix},\quad \boldsymbol{U}=\begin{bmatrix}1&0&1\\0&1&1\\0&0&1\end{bmatrix}.$$

(b) We first perform elimination to obtain

$$\boldsymbol{E}_{31}\boldsymbol{E}_{32}\boldsymbol{A}=\begin{bmatrix}1&0&0\\0&1&0\\-3&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}\begin{bmatrix}0&1&1\\1&0&1\\2&3&6\end{bmatrix}=\begin{bmatrix}0&1&1\\1&0&1\\0&0&1\end{bmatrix}=\boldsymbol{A}'.$$

Then we perform row exchange to obtain

$$\boldsymbol{P}_1^T\boldsymbol{A}'=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&1\\1&0&1\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&1\\0&1&1\\0&0&1\end{bmatrix}=\boldsymbol{U}.$$

Therefore, we can obtain $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{P}_1\boldsymbol{U}$ where

$$\boldsymbol{P}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{L} = \boldsymbol{E}_{32}^{-1}\boldsymbol{E}_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}, \quad \boldsymbol{U} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$