EE 2030 Linear Algebra — Chi-chao Chao
Spring 2012

Homework Assignment No. 2
Due 10:10am, March 28, 2012

`Reading:` Strang, Chapter 3.

`Problems for Solution:`

1. Suppose S and T are two subspaces of a vector space V. The *sum* $S+T$ contains all sums $\boldsymbol{s}+\boldsymbol{t}$ of a vector $\boldsymbol{s}$ in S and a vector $\boldsymbol{t}$ in T, i.e.,

$$S+T=\{\boldsymbol{s}+\boldsymbol{t} : \boldsymbol{s}\in S,\ \boldsymbol{t}\in T\}.$$

   The *intersection* $S\cap T$ contains all vectors in S and also in T, i.e.,

$$S\cap T=\{\boldsymbol{v} : \boldsymbol{v}\in S \text{ and } \boldsymbol{v}\in T\}.$$

   (a) Show that $S+T$ is a subspace of V.

   (b) Show that $S\cap T$ is a subspace of V.

2. True or false. (If it is true, prove it. Otherwise, find a counterexample.)

   (a) The subset $\{(a_1,a_2,a_3) : a_1+2a_2-3a_3=1\}$ of $\mathcal{R}^3$ is a subspace of $\mathcal{R}^3$.

   (b) Suppose $\boldsymbol{A}$ is an m by n real matrix. All the (m by 1) vectors $\boldsymbol{b}$ that are not in the column space $C(\boldsymbol{A})$ form a subspace of $\mathcal{R}^m$.

   (c) Matrices $\boldsymbol{A}$ and $\boldsymbol{B}=\boldsymbol{C}\boldsymbol{A}$ have the same nullspace when $\boldsymbol{C}$ is invertible.

3. Under what condition on b_1, b_2, b_3 is this system solvable? Find the complete solution when that condition holds:

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 4 & 9 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

4. Find the complete solution to

$$\begin{bmatrix} 1 & 1 & 1 & 1 & -3 \\ 2 & 3 & 1 & 4 & -9 \\ 1 & 1 & 1 & 2 & -5 \\ 2 & 2 & 2 & 3 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}=\begin{bmatrix} 6 \\ 17 \\ 8 \\ 14 \end{bmatrix}.$$

5. Prove that n vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ in an m-dimensional vector space V must be *linearly dependent* when $n > m$. (*Hint:* Let $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m$ form a basis for V. We can have

$$\boldsymbol{v}_j = \sum_{i=1}^{m} a_{ij}\boldsymbol{w}_i, \text{ for } j = 1, 2, \ldots, n.$$

Then consider

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_n\boldsymbol{v}_n = \mathbf{0}.$$

Should the values of $x_1, x_2, \ldots, x_n$ always be zero?)

6. Write down a matrix with the required property or explain why no such matrix exists.

   (a) The only solution to $\boldsymbol{Ax} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

   (b) Column space has basis $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$; nullspace has basis $\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$.

   (c) Column space contains $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$; row space contains $(1, 2)$ but not $(1, 3)$.

7. Find a basis for each of the four subspaces of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 3 & 5 \\ -1 & -3 & 1 & 0 \end{bmatrix}.$$

8. (a) For matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, show that $\operatorname{rank}(\boldsymbol{AB}) \leq \operatorname{rank}(\boldsymbol{B})$. (*Hint:* Argue that the rows of $\boldsymbol{AB}$ are linear combinations of the rows of $\boldsymbol{B}$.)

   (b) Also show that $\operatorname{rank}(\boldsymbol{AB}) \leq \operatorname{rank}(\boldsymbol{A})$. (*Hint:* Consider $\boldsymbol{B}^T\boldsymbol{A}^T$.)

   (c) Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are n by n matrices, and $\boldsymbol{AB} = \boldsymbol{I}$. Show that $\boldsymbol{A}$ is invertible and $\boldsymbol{B}$ must be its inverse. (*Hint:* First show that the rank of $\boldsymbol{A}$ is n by using $\operatorname{rank}(\boldsymbol{AB}) \leq \operatorname{rank}(\boldsymbol{A})$.)