Solution to Homework Assignment No. 2

1. (a) Let $\boldsymbol{x}_1 = \boldsymbol{s}_1 + \boldsymbol{t}_1 \in S + T$ and $\boldsymbol{x}_2 = \boldsymbol{s}_2 + \boldsymbol{t}_2 \in S + T$, where $\boldsymbol{s}_1, \boldsymbol{s}_2 \in S$ and $\boldsymbol{t}_1, \boldsymbol{t}_2 \in T$. We need to check the following two conditions:

- Consider $\boldsymbol{x}_1 + \boldsymbol{x}_2 = (\boldsymbol{s}_1 + \boldsymbol{t}_1) + (\boldsymbol{s}_2 + \boldsymbol{t}_2) = (\boldsymbol{s}_1 + \boldsymbol{s}_2) + (\boldsymbol{t}_1 + \boldsymbol{t}_2)$. Since $\boldsymbol{s}_1, \boldsymbol{s}_2 \in S$ and S is a subspace, we have $\boldsymbol{s}_1 + \boldsymbol{s}_2 \in S$. Similarly, we have $\boldsymbol{t}_1 + \boldsymbol{t}_2 \in T$. Therefore, $\boldsymbol{x}_1 + \boldsymbol{x}_2 = (\boldsymbol{s}_1 + \boldsymbol{s}_2) + (\boldsymbol{t}_1 + \boldsymbol{t}_2) \in S + T$.
- For any scalar c, consider $c\boldsymbol{x}_1 = c\boldsymbol{s}_1 + c\boldsymbol{t}_1$. Since $\boldsymbol{s}_1 \in S$, $\boldsymbol{t}_1 \in T$ and S, T are subspaces of V, we have $c\boldsymbol{s}_1 \in S$ and $c\boldsymbol{t}_1 \in T$. Therefore, $c\boldsymbol{x}_1 = c\boldsymbol{s}_1 + c\boldsymbol{t}_1 \in S + T$.

Therefore, $S + T$ is a subspace of the vector space V.

(b) Let $\boldsymbol{x}_1 \in S \cap T$. That is to say that $\boldsymbol{x}_1 \in S$ and $\boldsymbol{x}_1 \in T$. Also let $\boldsymbol{x}_2 \in S \cap T$. We have $\boldsymbol{x}_2 \in S$ and $\boldsymbol{x}_2 \in T$. We need to check the following two conditions:

- Consider $\boldsymbol{x}_1 + \boldsymbol{x}_2$. Since $\boldsymbol{x}_1, \boldsymbol{x}_2 \in S$ and S is a subspace of V, we have $\boldsymbol{x}_1 + \boldsymbol{x}_2 \in S$. Similarly, we have $\boldsymbol{x}_1 + \boldsymbol{x}_2 \in T$. Therefore, $\boldsymbol{x}_1 + \boldsymbol{x}_2 \in S \cap T$.
- For any scalar c, consider $c\boldsymbol{x}_1$. Since $\boldsymbol{x}_1 \in S$ and S is a subspaces of V, we have $c\boldsymbol{x}_1 \in S$. Similarly, we have $c\boldsymbol{x}_1 \in T$. Therefore, $c\boldsymbol{x}_1 \in S \cap T$.

Therefore, $S \cap T$ is a subspace of the vector space V.

2. (a) False. This subset is not a subspace of $\mathcal{R}^3$. Let $A = \{(a_1, a_2, a_3) : a_1 + 2a_2 - 3a_3 = 1\}$. Consider $(1,0,0)$, $(1,3,2) \in A$ and $(1,0,0) + (1,3,2) = (2,3,2)$. Since $2 + 2 \cdot 3 - 3 \cdot 2 = 2 \neq 1$, we know that $(2,3,2) \notin A$. Therefore, A is not a subspace of $\mathcal{R}^3$.

(b) False. All the vectors $\boldsymbol{b}$ that are not in the column space $\mathcal{C}(\boldsymbol{A})$ do not form a subspace of $\mathcal{R}^m$. Consider $\boldsymbol{A} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and we konw that $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ are not in the column space $\mathcal{C}(\boldsymbol{A})$. However, $\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is in the column space $\mathcal{C}(\boldsymbol{A})$. Therefore, all the vectors $\boldsymbol{b}$ that are not in the column space $\mathcal{C}(\boldsymbol{A})$ do not form a subspace of $\mathcal{R}^m$.

(c) Ture. Since

$$\begin{aligned} \mathcal{N}(\boldsymbol{B}) &= \{\boldsymbol{x} : \quad \boldsymbol{B}\boldsymbol{x} = \boldsymbol{0}\} \\ &= \{\boldsymbol{x} : \quad \boldsymbol{C}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\} \\ &= \{\boldsymbol{x} : \quad \boldsymbol{C}^{-1}\boldsymbol{C}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{C}^{-1}\boldsymbol{0}\} \quad \text{(because } \boldsymbol{C} \text{ is invertible)} \\ &= \{\boldsymbol{x} : \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\} \\ &= \mathcal{N}(\boldsymbol{A}) \end{aligned}$$

matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same nullspace when $\boldsymbol{C}$ is invertible.

3. We can solve this system by the following procedure:

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 4 & 9 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & -2b_1 + b_2 \\ 4 & 9 & -8 & b_3 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & -2b_1 + b_2 \\ 0 & 1 & 0 & -4b_1 + b_3 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & -2b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 - b_2 + b_3 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 5b_1 - 2b_2 \\ 0 & 1 & 0 & -2b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 - b_2 + b_3 \end{array}\right].$$

The system is solvable if $-2b_1 - b_2 + b_3 = 0$, i.e.,

$$b_3 = 2b_1 + b_2.$$

When the above condition holds, we need to solve

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 5b_1 - 2b_2 \\ 0 & 1 & 0 & -2b_1 + b_2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The pivot variables are x_1 and x_2, and the free variable is x_3. First, we want to find a particular solution. Choose the free variables $x_3 = 0$. Then we have a particular solution given by

$$\boldsymbol{x}_p = \begin{bmatrix} 5b_1 - 2b_2 \\ -2b_1 + b_2 \\ 0 \end{bmatrix}.$$

Then we want to find the nullspace vectors x_n. Given $x_3 = 1$, we can have $(x_1, x_2) = (2, 0)$. Therefore, we can obtain

$$\boldsymbol{x}_n = x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

where $x_3 \in \mathcal{R}$. Finally, the complete solution is

$$\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n = \begin{bmatrix} 5b_1 - 2b_2 \\ -2b_1 + b_2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

where $x_3 \in \mathcal{R}$ if $b_3 = 2b_1 + b_2$.

4. Consider the augmented matrix and perform elimination, and we have

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{array}\right] \Longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -2 & 3 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The pivot variables are x_1, x_2, and x_4, and the free variables are x_3 and x_5. First, we want to find a particular solution. Choose the free variables as $x_3 = x_5 = 0$. Then we have $x_1 = 3$, $x_2 = 1$, and $x_4 = 2$. Therefore, a particular solution is

$$\boldsymbol{x}_p = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}.$$

Then we want to find the nullspace vectors $\boldsymbol{x}_n$.

- Given $(x_3, x_5) = (1, 0)$, we can have $(x_1, x_2, x_4) = (-2, 1, 0)$.
- Given $(x_3, x_5) = (0, 1)$, we can have $(x_1, x_2, x_4) = (2, -1, 2)$.

Therefore, we can obtain

$$\boldsymbol{x}_n = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

where $x_3, x_5 \in \mathcal{R}$. Finally, the complete solution is given by

$$\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

where $x_3, x_5 \in \mathcal{R}$.

5. Let $\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_m$ form a basis for $\boldsymbol{V}$. We can have

$$\boldsymbol{v}_j = \sum_{i=1}^{m} a_{ij} \boldsymbol{w}_i, \quad \text{for } j = 1, 2, \cdots n.$$

Then consider the following equation:

$$\begin{aligned} & x_1 \boldsymbol{v}_1 + x_2 \boldsymbol{v}_2 + \cdots + x_n \boldsymbol{v}_n = \mathbf{0} \\ \Longrightarrow\ & x_1(a_{11}\boldsymbol{w}_1 + a_{21}\boldsymbol{w}_2 + \cdots + a_{m1}\boldsymbol{w}_m) + x_2(a_{12}\boldsymbol{w}_1 + a_{22}\boldsymbol{w}_2 + \cdots + a_{m2}\boldsymbol{w}_m) + \\ & \qquad \cdots + x_n(a_{1n}\boldsymbol{w}_1 + a_{2n}\boldsymbol{w}_2 + \cdots + a_{mn}\boldsymbol{w}_m) = \mathbf{0} \\ \Longrightarrow\ & (x_1 a_{11} + x_2 a_{12} + \cdots + x_n a_{1n})\boldsymbol{w}_1 + (x_1 a_{21} + x_2 a_{22} + \cdots + x_n a_{2n})\boldsymbol{w}_2 + \\ & \qquad \cdots + (x_1 a_{m1} + x_2 a_{m2} + \cdots + x_n a_{mn})\boldsymbol{w}_m = \mathbf{0}. \end{aligned}$$

Since $\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_m$ form a basis, they are linearly independent. We know the only solution to the above equation is

$$\Longrightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$
$$\Longrightarrow \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}.$$

Since $n > m$, we have $r \leq m < n$. There are $n - r > 0$ free variables and hence there exist nonzero solutions $\boldsymbol{x}$. Therefore, $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ must be linearly dependent.

6. (a) Since $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is 3 by 1 and $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is 2 by 1, we know that $\boldsymbol{A}$ is a 3 by 2 matrix. For $\boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to be the only solution to $\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, the nullspace of $\boldsymbol{A}$ must contain the zero vector only. Hence, the rank of $\boldsymbol{A}$ should be 2. Let $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix}$, where $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are column vectors. We have

$$\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

which gives

$$\boldsymbol{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

And $\boldsymbol{a}_1$ can be any 3×1 column vector which is not a multiple of $\boldsymbol{a}_1$. For example, we can choose

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

(b) No such matrix exists. Since the column space and nullspace both have three components, the desired matrix is 3 by 3, say $\boldsymbol{B}$. We can find $\dim(\mathcal{N}(\boldsymbol{B})) = 1 \neq 2 = 3 - 1 = 3 - \text{rank}(\boldsymbol{B})$, which is not possible. Therefore, no such matrix exists.

(c) No such matrix exists. It is clear that the desired matrix is 3 by 2. Since the column space contains $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and these two vectors are linearly independent, we know that the rank of the desired matrix must be 2. It follows that the dimension of the row space is 2 and thus the row space should be $\mathcal{R}^2$. Therefore, $(1, 3)$ has to be in the row space.

7. Convert $\boldsymbol{A}$ into the RRE form:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 3 & 5 \\ -1 & -3 & 1 & 0 \end{bmatrix} \implies \boldsymbol{R} = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, a basis for the row space of $\boldsymbol{A}$ can be given by

$$(1, 3, 0, 1), \ (0, 0, 1, 1).$$

The pivot columns are the 1st and 3rd columns of $\boldsymbol{R}$, and hence a basis for the column space of $\boldsymbol{A}$ can be given by

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

Since x_1 and x_3 are pivot variables and x_2 and x_4 are free variables, a basis for the nullspace can be given by the special solutions:

$$\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

We can have $\boldsymbol{R} = \boldsymbol{E}\boldsymbol{A}$ where

$$\boldsymbol{E} = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 0 \\ 5 & -2 & 1 \end{bmatrix}.$$

Since the last row of $\boldsymbol{R}$ is a zero row, a basis for the left nullspace can be given by the last row of $\boldsymbol{E}$:

$$(5, -2, 1).$$

8. (a) Let $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}$ where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are m by n, n by l, and m by l, respectively, with $\boldsymbol{a}_k$, $\boldsymbol{b}_k$, and $\boldsymbol{c}_k$ denoting the kth row of $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, respectively. We have

$$\boldsymbol{c}_i = \boldsymbol{a}_i \boldsymbol{B} = \sum_{j=1}^{n} a_{ij} \boldsymbol{b}_j \quad \text{for} \quad 1 \le i \le m$$

which shows that the rows of $\boldsymbol{C}$ are linear combinations of the rows of $\boldsymbol{B}$. Hence, any linear combination of the rows of $\boldsymbol{C}$ is a linear combination of the rows of $\boldsymbol{B}$, which yields

$$\mathcal{C}(\boldsymbol{C}^T) \subseteq \mathcal{C}(\boldsymbol{B}^T).$$

The rank of $\boldsymbol{C}$ is the maximal number of linearly independent vectors in $\mathcal{C}(\boldsymbol{C}^T)$, which in turn cannot exceed the maximal number of linearly independent vectors in $\mathcal{C}(\boldsymbol{B}^T)$, i.e., the rank of $\boldsymbol{B}$. Therefore,

$$\operatorname{rank}(\boldsymbol{A}\boldsymbol{B}) = \operatorname{rank}(\boldsymbol{C}) \le \operatorname{rank}(\boldsymbol{B}).$$

(b) We can obtain $\boldsymbol{C}^T = \boldsymbol{B}^T \boldsymbol{A}^T$ by taking transposition on both sides of $\boldsymbol{C} = \boldsymbol{AB}$. It now follows from (a) that

$$\text{rank}(\boldsymbol{C}^T) \leq \text{rank}(\boldsymbol{A}^T).$$

Together with the fact that

$$\text{rank}(\boldsymbol{C}) = \text{rank}(\boldsymbol{C}^T) \quad \text{and} \quad \text{rank}(\boldsymbol{A}^T) = \text{rank}(\boldsymbol{A})$$

we finally arrive at

$$\text{rank}(\boldsymbol{AB}) = \text{rank}(\boldsymbol{C}) = \text{rank}(\boldsymbol{C}^T) \leq \text{rank}(\boldsymbol{A}^T) = \text{rank}(\boldsymbol{A}).$$

(c) Here $\boldsymbol{A}$ and $\boldsymbol{B}$ are n by n matrices. From (b), we can obtain

$$n = \text{rank}(\boldsymbol{I}) = \text{rank}(\boldsymbol{AB}) \leq \text{rank}(\boldsymbol{A}).$$

Since $\boldsymbol{A}$ is an n by n matrix, we have $\text{rank}(\boldsymbol{A}) = n$. It follows that $\boldsymbol{A}$ is nonsingular and hence invertible. Now since $\boldsymbol{A}$ is invertible and $\boldsymbol{AB} = \boldsymbol{I}$, by part (a) of Problem 3 in Homework Assignment No. 1, we can have $\boldsymbol{B} = \boldsymbol{A}^{-1}$.