Solution to Homework Assignment No. 3

1. Assume both systems have solutions. We can have

$$\begin{aligned} \boldsymbol{Ax} &= \boldsymbol{b} \\ \Rightarrow \quad \boldsymbol{x}^T\boldsymbol{A}^T &= \boldsymbol{b}^T \\ \Rightarrow \quad \boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{y} &= \boldsymbol{b}^T\boldsymbol{y} \\ \Rightarrow \quad \boldsymbol{x}^T\boldsymbol{0} &= \boldsymbol{b}^T\boldsymbol{y} \\ \Rightarrow \quad 0 &= \boldsymbol{y}^T\boldsymbol{b} \end{aligned}$$

which contradicts $\boldsymbol{y}^T\boldsymbol{b} \neq 0$.

2. (a) Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

and the plane spanned by the vectors $(1, 1, 2)$ and $(1, 2, 3)$ is

$$V = \{\boldsymbol{v} : \boldsymbol{v} = a_1(1, 1, 2) + a_2(1, 2, 3), \forall a_1, a_2 \in \mathcal{R}\} = \mathcal{C}(\boldsymbol{A}^T).$$

The orthogonal complement of V is hence the nullspace of $\boldsymbol{A}$. The RRE form of $\boldsymbol{A}$ can be given by

$$\boldsymbol{R}_A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

We can therefore find a basis for $\mathcal{N}(\boldsymbol{A})$ as $(-1, -1, 1)$. As a result, we can have

$$V^\perp = \mathcal{N}(\boldsymbol{A}) = \{\boldsymbol{w} : \boldsymbol{w} = a_3(-1, -1, 1), \forall a_3 \in \mathcal{R}\}.$$

(b) It is equivalent to finding a homogeneous equation whose solution space is V. Let $\boldsymbol{B} = [-1 \ \ -1 \ \ 1]$ and then the homogeneous equation $\boldsymbol{Bx} = 0$ where $\boldsymbol{x} = [x_1 \ \ x_2 \ \ x_3]^T$ gives $-x_1 - x_2 + x_3 = 0$. From (a), we have $V^\perp = \mathcal{N}(\boldsymbol{A}) = \mathcal{C}(\boldsymbol{B}^T)$. Therefore, the solution space of $\boldsymbol{Bx} = 0$ is

$$\mathcal{N}(\boldsymbol{B}) = \mathcal{C}(\boldsymbol{B}^T)^\perp = \mathcal{N}(\boldsymbol{A})^\perp = \mathcal{C}(\boldsymbol{A}^T) = V.$$

3. (a) The projection matrix $\boldsymbol{P}$ onto the column space of $\boldsymbol{A}$ can be obtained as

$$\begin{aligned} \boldsymbol{P} &= \boldsymbol{A}(\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T \\ &= \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 4 \end{bmatrix} \\ &= \frac{1}{9} \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & -2 \\ 2 & -2 & 8 \end{bmatrix}. \end{aligned}$$

(b) From the projection matrix $\boldsymbol{P}$ derived in (a), we can have

$$\boldsymbol{x}_c = \boldsymbol{P}\boldsymbol{x} = \frac{1}{9}\begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & -2 \\ 2 & -2 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}.$$

And hence

$$\boldsymbol{x}_{ln} = \boldsymbol{x} - \boldsymbol{x}_c = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}.$$

4. (a) Let

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \boldsymbol{x}_1 = \begin{bmatrix} C_1 \\ D_1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix}.$$

The best least squares straight line fit can be obtained by solving $\boldsymbol{A}_1^T\boldsymbol{A}_1\boldsymbol{x}_1 = \boldsymbol{A}_1^T\boldsymbol{b}$. Hence we can have

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} C_1 \\ D_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} C_1 \\ D_1 \end{bmatrix} = \begin{bmatrix} -6 \\ -15 \end{bmatrix}$$

$$\Longrightarrow C_1 = \frac{-3}{10}, \quad D_1 = \frac{-12}{5}.$$

As a result, the best least squares straight line fit is

$$b = \frac{-3}{10} - \frac{12}{5}t.$$

(b) Let

$$\boldsymbol{A}_2 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} C_2 \\ D_2 \\ E_2 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix}.$$

The best least squares parabola fit can be obtained by solving $\boldsymbol{A}_2^T\boldsymbol{A}_2\boldsymbol{x}_2 =$

$\boldsymbol{A}_2^T\boldsymbol{b}$. Then we can have

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} C_2 \\ D_2 \\ E_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix} \begin{bmatrix} C_2 \\ D_2 \\ E_2 \end{bmatrix} = \begin{bmatrix} -6 \\ -15 \\ -21 \end{bmatrix}$$

$$\Rightarrow C_2 = \frac{-3}{10}, \quad D_2 = \frac{-12}{5}, \quad E_2 = 0.$$

As a result, the best least squares parabola fit is

$$b = \frac{-3}{10} - \frac{12}{5}t$$

the same as the best least squares straight line fit.

5. (a) We can have

$$\begin{aligned} \boldsymbol{Q}^T\boldsymbol{Q} &= \left(\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^T\right)^T\left(\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^T\right) \\ &= \left(\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^T\right)\left(\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^T\right) \\ &= \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^T - 2\boldsymbol{u}\boldsymbol{u}^T + 4\boldsymbol{u}\boldsymbol{u}^T\boldsymbol{u}\boldsymbol{u}^T \\ &= \boldsymbol{I} - 4\boldsymbol{u}\boldsymbol{u}^T + 4\boldsymbol{u}\boldsymbol{u}^T \\ &= \boldsymbol{I} \end{aligned}$$

where the fourth equality follows from the fact that $\boldsymbol{u}$ is an unit vector and $\boldsymbol{u}^T\boldsymbol{u} = 1$. Therefore, $\boldsymbol{Q}$ is an orthogonal matrix.

(b) Since $\boldsymbol{Q}^T = \left(\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^T\right)^T = \boldsymbol{I}^T - 2(\boldsymbol{u}\boldsymbol{u}^T)^T = \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^T = \boldsymbol{Q}$, we can have $\boldsymbol{Q}^2 = \boldsymbol{Q}\boldsymbol{Q} = \boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}$.

(c) By the definition of $\boldsymbol{Q}$, we can have

$$\begin{aligned} \boldsymbol{Q}_1 = \boldsymbol{I} - 2\boldsymbol{u}_1\boldsymbol{u}_1^T &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ \boldsymbol{Q}_2 = \boldsymbol{I} - 2\boldsymbol{u}_2\boldsymbol{u}_2^T &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 0 \\ \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}. \end{aligned}$$

Since

$$\begin{aligned} \boldsymbol{Q}_1^T\boldsymbol{Q}_1 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \boldsymbol{Q}_2^T\boldsymbol{Q}_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are orthogonal matrices.

6. (a) Let $\boldsymbol{A} = [\boldsymbol{a}_1\ \boldsymbol{a}_2]$ where

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \boldsymbol{a}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

By applying the Gram-Schmit process, we can have:

(i) $$\boldsymbol{A}_1 = \boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \|\boldsymbol{A}_1\|^2 = \boldsymbol{A}_1^T \boldsymbol{A}_1 = 9$$

$$\Longrightarrow \boldsymbol{q}_1 = \frac{\boldsymbol{A}_1}{\|\boldsymbol{A}_1\|} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

(ii) $$\boldsymbol{A}_2 = \boldsymbol{a}_2 - \left(\boldsymbol{q}_1^T \boldsymbol{a}_2\right)\boldsymbol{q}_1$$

$$= \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - \left(\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}\right)\frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\|\boldsymbol{A}_2\|^2 = \boldsymbol{A}_2^T \boldsymbol{A}_2 = 2 \quad \Longrightarrow \boldsymbol{q}_2 = \frac{\boldsymbol{A}_2}{\|\boldsymbol{A}_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Therefore, $\{\boldsymbol{q}_1, \boldsymbol{q}_2\}$ is an orthonormal basis for the column space of $\boldsymbol{A}$.

(b) From (a), we can express $\boldsymbol{A}$ as

$$\begin{aligned} \boldsymbol{A} &= \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{q}_1^T \boldsymbol{a}_1 & \boldsymbol{q}_1^T \boldsymbol{a}_2 \\ 0 & \boldsymbol{q}_2^T \boldsymbol{a}_2 \end{bmatrix} \\ &= \begin{bmatrix} 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix}. \end{aligned}$$

Hence we can have

$$\boldsymbol{Q} = \begin{bmatrix} 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \end{bmatrix}, \quad \boldsymbol{R} = \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix}.$$

(c) The projection matrix $\boldsymbol{P}$ onto the column space of $\boldsymbol{A}$ can be derived as

$$\begin{aligned} \boldsymbol{P} &= \boldsymbol{A}\left(\boldsymbol{A}^T\boldsymbol{A}\right)^{-1}\boldsymbol{A}^T \\ &= \boldsymbol{Q}\boldsymbol{R}\left(\boldsymbol{R}^T\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{R}\right)^{-1}\boldsymbol{R}^T\boldsymbol{Q}^T \\ &= \boldsymbol{Q}\boldsymbol{R}\left(\boldsymbol{R}^T\boldsymbol{R}\right)^{-1}\boldsymbol{R}^T\boldsymbol{Q}^T \\ &= \boldsymbol{Q}\boldsymbol{R}\boldsymbol{R}^{-1}\left(\boldsymbol{R}^T\right)^{-1}\boldsymbol{R}^T\boldsymbol{Q}^T \\ &= \boldsymbol{Q}\boldsymbol{Q}^T \\ &= \begin{bmatrix} 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \\ &= \frac{1}{18}\begin{bmatrix} 2 & 4 & 4 \\ 4 & 17 & -1 \\ 4 & -1 & 17 \end{bmatrix}. \end{aligned}$$

Therefore, the projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$ is

$$\boldsymbol{P}\boldsymbol{b} = \frac{1}{18}\begin{bmatrix} 2 & 4 & 4 \\ 4 & 17 & -1 \\ 4 & -1 & 17 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 7 \\ 5 \\ 23 \end{bmatrix}.$$

(d) The least squares solution $\hat{\boldsymbol{x}}$ to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{c}$ can be obtained by solving $\boldsymbol{A}^T\boldsymbol{A}\hat{\boldsymbol{x}} = \boldsymbol{A}^T\boldsymbol{c}$. Hence we can have

$$\begin{aligned} &\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix}\hat{\boldsymbol{x}} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ \Longrightarrow\ &\begin{bmatrix} 9 & 9 \\ 9 & 11 \end{bmatrix}\hat{\boldsymbol{x}} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \\ \Longrightarrow\ &\hat{\boldsymbol{x}} = \begin{bmatrix} 5/9 \\ 0 \end{bmatrix}. \end{aligned}$$

7. (a) Let $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^2$. By applying the Gram-Schmidt process, we can have:

(i) $F_1(x) = f_1(x) = 1, \quad \|F_1(x)\|^2 = \langle F_1(x), F_1(x)\rangle = \int_{-2}^{2} 1 \cdot 1\, dx = 4$

$$\Longrightarrow q_1(x) = \frac{F_1(x)}{\|F_1(x)\|} = \frac{1}{2}.$$

(ii) $F_2(x) = f_2(x) - \langle q_1(x), f_2(x)\rangle q_1(x) = x - \left(\int_{-2}^{2}\frac{1}{2}x dx\right)\frac{1}{2} = x$

$$\|F_2(x)\|^2 = \langle F_2(x), F_2(x)\rangle = \int_{-2}^{2} x \cdot x dx = \frac{16}{3}$$

$$\Longrightarrow q_2(x) = \frac{F_2(x)}{\|F_2(x)\|} = \frac{\sqrt{3}}{4}x.$$

(iii) $F_3(x) = f_3(x) - \langle q_1(x), f_3(x)\rangle q_1(x) - \langle q_2(x), f_3(x)\rangle q_2(x)$

$$= x^2 - \left(\int_{-2}^{2} \frac{1}{2}x^2 dx\right)\frac{1}{2} - \left(\int_{-2}^{2} \frac{\sqrt{3}}{4}x \cdot x^2 dx\right)\frac{\sqrt{3}}{4}x = x^2 - \frac{4}{3}$$

$$\|F_3(x)\|^2 = \langle F_3(x), F_3(x)\rangle = \int_{-2}^{2}\left(x^2 - \frac{4}{3}\right)\left(x^2 - \frac{4}{3}\right)dx = \frac{256}{45}$$

$$\Longrightarrow q_3(x) = \frac{F_3(x)}{\|F_3(x)\|} = \frac{3\sqrt{5}}{16}x^2 - \frac{\sqrt{5}}{4}.$$

Therefore, $\{q_1(x), q_2(x), q_3(x)\}$ forms an orthonormal basis for the subspace spanned by 1, x, and x^2.

(b) Since

$$\langle x^2 + 2x, q_1(x)\rangle = \frac{8}{3}$$
$$\langle x^2 + 2x, q_2(x)\rangle = \frac{8\sqrt{3}}{3}$$
$$\langle x^2 + 2x, q_3(x)\rangle = \frac{16\sqrt{5}}{15}$$

we can express $x^2 + 2x$ as

$$\begin{aligned} & x^2 + 2x \\ = \; & \langle x^2 + 2x, q_1(x)\rangle q_1(x) + \langle x^2 + 2x, q_2(x)\rangle q_2(x) + \langle x^2 + 2x, q_3(x)\rangle q_3(x) \\ = \; & \frac{8}{3}\cdot\frac{1}{2} + \frac{8\sqrt{3}}{3}\cdot\frac{\sqrt{3}}{4}x + \frac{16\sqrt{5}}{15}\cdot\left(\frac{3\sqrt{5}}{16}x^2 - \frac{\sqrt{5}}{4}\right). \end{aligned}$$

8. (a) Let

$$f_1(t) = \frac{\cos t}{\sqrt{\int_{-\pi}^{\pi}\cos^2 t dt}} = \frac{\cos t}{\sqrt{\pi}}, \quad f_2(t) = \frac{\sin t}{\sqrt{\int_{-\pi}^{\pi}\sin^2 t dt}} = \frac{\sin t}{\sqrt{\pi}}.$$

Then $f_1(t)$, $f_2(t)$ are orthonormal functions. The projection of $f(t) = \sin 2t$ onto the subspace spanned by $f_1(t)$ and $f_2(t)$ is given by

$$\langle f_1(t), f(t)\rangle f_1(t) + \langle f_2(t), f(t)\rangle f_2(t)$$

where

$$\langle f_1(t), f(t)\rangle = \int_{-\pi}^{\pi}\frac{\cos t}{\sqrt{\pi}}\cdot\sin 2t dt = 0$$
$$\langle f_2(t), f(t)\rangle = \int_{-\pi}^{\pi}\frac{\sin t}{\sqrt{\pi}}\cdot\sin 2t dt = 0.$$

Therefore, the closest function $a\cos t + b\sin t$ to $\sin 2t$ is

$$0\cdot\frac{\cos t}{\sqrt{\pi}} + 0\cdot\frac{\sin t}{\sqrt{\pi}} = 0.$$

(b) Let

$$g_1(t) = \frac{1}{\sqrt{\int_{-\pi}^{\pi} 1^2 dt}} = \frac{1}{\sqrt{2\pi}}, \quad g_2(t) = \frac{t}{\sqrt{\int_{-\pi}^{\pi} t^2 dt}} = \sqrt{\frac{3}{2\pi^3}}\, t.$$

Then $g_1(t)$, $g_2(t)$ are orthonormal functions. The projection of $f(t) = \sin 2t$ onto the subspace spanned by $g_1(t)$ and $g_2(t)$ is given by

$$\langle g_1(t), f(t)\rangle g_1(t) + \langle g_2(t), f(t)\rangle g_2(t)$$

where

$$\langle g_1(t), f(t)\rangle = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} \cdot \sin 2t dt = 0$$
$$\langle g_2(t), f(t)\rangle = \int_{-\pi}^{\pi} \sqrt{\frac{3}{2\pi^3}}\, t \cdot \sin 2t dt = -\sqrt{\frac{3}{2\pi}}.$$

Therefore, the closest function $c + dt$ to $\sin 2t$ is

$$0 \cdot \frac{1}{\sqrt{2}} + \left(-\sqrt{\frac{3}{2\pi}}\right) \cdot \sqrt{\frac{3}{2\pi^3}}\, t = -\frac{3}{2\pi^2}\, t.$$