EE 2030 Linear Algebra — Chi-chao Chao
Spring 2012

Homework Assignment No. 4
Due 10:10am, May 2, 2012

`Reading:` Strang, Chapter 5.

`Problems for Solution:`

1. (*This problem counts double.*) Find the determinants of

   (a) a rank one matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 \\ 4 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 & 7 \end{bmatrix};$$

   (b) the upper triangular matrix

$$\boldsymbol{U} = \begin{bmatrix} 4 & 4 & 8 & 8 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 2 \end{bmatrix};$$

   (c) the lower triangular matrix $\boldsymbol{U}^T$;

   (d) the inverse matrix $\boldsymbol{U}^{-1}$;

   (e) the "reverse triangular" matrix that results from row exchanges

$$\boldsymbol{M} = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 1 & 2 & 2 \\ 4 & 4 & 8 & 8 \end{bmatrix};$$

   (f) the $-1, 1, 1$ tridiagonal matrix

$$\boldsymbol{F} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

2. (a) Use row operations to verify that the 3 by 3 "Vandermonde determinant" is

$$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b-a)(c-a)(c-b).$$

(b) Show that an n by n skew-symmetric matrix $\boldsymbol{A}$ has $\det \boldsymbol{A} = 0$ if n is odd. (Note that a skew-symmetric matrix satisfies $\boldsymbol{A}^T = -\boldsymbol{A}$.)

3. (a) Find all the *odd* permutations of the numbers $\{1, 2, 3, 4\}$. These are the permutations coming from an odd number of exchanges and leading to $\det \boldsymbol{P}_\sigma = -1$.

   (b) The circular shift permutes $(1, 2, \ldots, n-1, n)$ into $(2, 3, \ldots, n, 1)$. What is the corresponding matrix $\boldsymbol{P}_\sigma$ and (depending on n) what is its determinant?

4. Consider the n by n matrix $\boldsymbol{A}_n$ with zeros on the diagonal and ones elsewhere:

$$\boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{A}_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{A}_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad \ldots$$

   (a) Find the determinants of $\boldsymbol{A}_2$, $\boldsymbol{A}_3$, and $\boldsymbol{A}_4$.

   (b) What is $\det \boldsymbol{A}_n$? (*Hint:* Start by adding all rows (except the last) to the last row, and then factoring out a constant.)

5. The matrix $\boldsymbol{B}_n$ is the $-1, 2, -1$ tridiagonal matrix $\boldsymbol{A}_n$ considered in class except that $b_{11} = 1$ instead of $a_{11} = 2$:

$$\boldsymbol{B}_2 = \begin{bmatrix} \mathbf{1} & -1 \\ -1 & 2 \end{bmatrix}, \quad \boldsymbol{B}_3 = \begin{bmatrix} \mathbf{1} & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad \boldsymbol{B}_4 = \begin{bmatrix} \mathbf{1} & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \quad \ldots$$

   (a) Show that $|\boldsymbol{B}_n| = a|\boldsymbol{B}_{n-1}| + b|\boldsymbol{B}_{n-2}|$, for $n \geq 4$. Find the constants a and b.

   (b) Find $|\boldsymbol{B}_2|, |\boldsymbol{B}_3|, |\boldsymbol{B}_4|$. Guess a formula for $|\boldsymbol{B}_n|$ and verify your result.

   (c) Alternatively, use the linearity in the first row of $\boldsymbol{B}_n$, where $\begin{bmatrix} 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ illustrated for $n = 3$, to show that $|\boldsymbol{B}_n| = |\boldsymbol{A}_n| - |\boldsymbol{A}_{n-1}|$. Then get a formula for $|\boldsymbol{B}_n|$ from the formula for $|\boldsymbol{A}_n|$ obtained in class.

6. (a) Use Cramer's rule to solve

$$\begin{aligned} x + 4y - z &= 1 \\ x + y + z &= 0 \\ 2x \quad + 3z &= 0. \end{aligned}$$

   (b) Use the cofactor matrix to find the inverse of

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{bmatrix}.$$

7. (a) The *Hadamard matrix* $\boldsymbol{H}$ has orthogonal rows. (The box is a *hypercube*.)

$$\text{What is } |\boldsymbol{H}| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix} = \text{volume of a hypercube in } \mathcal{R}^4?$$

(b) If the columns of a 4 by 4 matrix have lengths L_1, L_2, L_3, L_4, what is the largest possible value for the determinant (based on volume)? If all the entries of the matrix are 1 or -1, what are these lengths L_1, L_2, L_3, L_4 and what is the maximum determinant?