EE 2030 Linear Algebra Chi-chao Chao
Spring 2012

Homework Assignment No. 5
Due 10:10am, June 1, 2012

`Reading:` Strang, Sections 6.1–6.6, Handout "Spectral Theorem."

`Problems for Solution:`

1. Suppose λ is an eigenvalue of an invertible matrix $\boldsymbol{A}$ and $\boldsymbol{x}$ is the associated eigenvector.
   (a) Show that this same $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}+\boldsymbol{I}$, and find the corresponding eigenvalue.
   (b) Assuming $\lambda \neq 0$, show that $\boldsymbol{x}$ is also an eigenvector of $\boldsymbol{A}^{-1}$, and find the corresponding eigenvalue.
2. Determine if each of the following matrices is diagonalizable. If it is, find an invertible matrix $\boldsymbol{S}$ and a diagonal matrix $\boldsymbol{\Lambda}$ such that $\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = \boldsymbol{\Lambda}$.
   (a) $\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.
   (b) $\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$.
3. Substitute $\boldsymbol{A} = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^{-1}$ into the product $(\lambda_1\boldsymbol{I}-\boldsymbol{A})(\lambda_2\boldsymbol{I}-\boldsymbol{A})\cdots(\lambda_n\boldsymbol{I}-\boldsymbol{A})$ and explain why this produces the zero matrix. We are substituting the matrix $\boldsymbol{A}$ for the variable λ in the characteristic polynomial $p(\lambda) = \det(\boldsymbol{A}-\lambda\boldsymbol{I})$. (The **Cayley-Hamilton Theorem** says that this product is always $p(\boldsymbol{A}) =$ *zero matrix*, even if $\boldsymbol{A}$ is not diagonalizable.)
4. (a) Consider the homogeneous difference equation:
   $$M_{k+2} + 3M_{k+1} + 2M_k = 0, \quad k \geq 0$$
   subject to $M_0 = 0$ and $M_1 = 1$. Find a general formula for M_k, $k \geq 0$.
   (b) Consider the homogeneous differential equation:
   $$u'' + 3u' + 2u = 0$$
   subject to $u(0) = 0$ and $u'(0) = 1$. Find a general formula for $u(t)$.
5. Suppose $\boldsymbol{A}$ is a real skew-symmetric matrix, i.e., $\boldsymbol{A}^T = -\boldsymbol{A}$.
   (a) Show that $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = 0$ for every real vector $\boldsymbol{x}$.

(b) Show that the eigenvalues of $\boldsymbol{A}$ are pure imaginary.

(c) Show that the determinant of $\boldsymbol{A}$ is positive or zero (not negative).

6. Consider

$$\boldsymbol{A} = \begin{bmatrix} 10 & -6 \\ -6 & 10 \end{bmatrix}.$$

(a) Find the $\boldsymbol{Q\Lambda Q}^T$ decomposition of $\boldsymbol{A}$, where $\boldsymbol{Q}$ is orthogonal and $\boldsymbol{\Lambda}$ is diagonal.

(b) Find the Cholesky ($\boldsymbol{CC}^T$) decomposition of $\boldsymbol{A}$, where $\boldsymbol{C}$ is lower triangular with positive diagonal entries.

7. Decide whether the following matrices are positive definite, negative definite, semidefinite, or indefinite:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 6 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 3 \end{bmatrix}, \quad \boldsymbol{C} = -\boldsymbol{B}, \quad \boldsymbol{D} = \boldsymbol{A}^{-1}.$$

8. Find the Jordan forms of $\boldsymbol{A}$ and $\boldsymbol{B}$ if

$$\boldsymbol{A} = \begin{bmatrix} -3 & 3 & -2 \\ -7 & 6 & -3 \\ 1 & -1 & 2 \end{bmatrix} \text{ and } \boldsymbol{B} = \begin{bmatrix} 0 & -1 & -1 \\ -3 & -1 & -2 \\ 7 & 5 & 6 \end{bmatrix}.$$

Is $\boldsymbol{A}$ similar to $\boldsymbol{B}$?