Solution to Homework Assignment No. 5

1. (a) Since $\boldsymbol{Ax} = \lambda\boldsymbol{x}$, we have

$$(\boldsymbol{A}+\boldsymbol{I})\boldsymbol{x} = \boldsymbol{Ax}+\boldsymbol{x} = \lambda\boldsymbol{x}+\boldsymbol{x} = (\lambda+1)\boldsymbol{x}.$$

Therefore, $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}+\boldsymbol{I}$ and the corresponding eigenvalue is $\lambda+1$.

(b) Since $\boldsymbol{Ax} = \lambda\boldsymbol{x}$, we have

$$\begin{aligned} & \boldsymbol{A}^{-1}\boldsymbol{Ax} = \lambda\boldsymbol{A}^{-1}\boldsymbol{x} \\ \Longrightarrow\ & \boldsymbol{x} = \lambda\boldsymbol{A}^{-1}\boldsymbol{x} \\ \overset{\lambda\neq 0}{\Longrightarrow}\ & \boldsymbol{A}^{-1}\boldsymbol{x} = \frac{1}{\lambda}\boldsymbol{x}. \end{aligned}$$

Therefore, $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}^{-1}$ and the corresponding eigenvalue is $1/\lambda$.

2. (a) Consider

$$\begin{aligned} \det(\boldsymbol{A}-\lambda\boldsymbol{I}) &= \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} \\ &= -\lambda^3+3\lambda^2 \\ &= -\lambda^2(\lambda-3) = 0. \end{aligned}$$

Thus, we have $\lambda = 0, 0, 3$. For $\lambda_1 = 0$, the AM of λ_1 equals 2. Besides,

$$\boldsymbol{A}-\lambda_1\boldsymbol{I} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the corresponding eigenvectors are

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

The GM of λ_1 is 2, which is equal to the AM of λ_1. For $\lambda_2 = 3$,

$$\boldsymbol{A}-\lambda_2\boldsymbol{I} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

and the corresponding eigenvector is

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The GM of λ_2 is 1, which is equal to the AM of λ_2. Therefore, $\boldsymbol{A}$ is diagonalizable with

$$\boldsymbol{S} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } \boldsymbol{\Lambda} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(b) Consider

$$\begin{aligned} \det(\boldsymbol{A} - \lambda \boldsymbol{I}) &= \begin{vmatrix} 2-\lambda & -1 & 1 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 2-\lambda \end{vmatrix} \\ &= (2-\lambda)^3 = 0. \end{aligned}$$

Thus, we have $\lambda = 2, 2, 2$. It can be seen that the AM of λ equals 3. Besides, for $\lambda = 2$,

$$\boldsymbol{A} - \lambda \boldsymbol{I} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and the corresponding eigenvector is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The GM of λ is 1, which is smaller than the AM of λ. As a result, $\boldsymbol{A}$ is not diagonalizable.

3. Substituting $\boldsymbol{A} = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^{-1}$, we can have

$$\begin{aligned} \lambda_j \boldsymbol{I} - \boldsymbol{A} &= \lambda_j \boldsymbol{I} - \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^{-1} \\ &= \boldsymbol{S}\lambda_j \boldsymbol{I}\boldsymbol{S}^{-1} - \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^{-1} \\ &= \boldsymbol{S}(\lambda_j \boldsymbol{I} - \boldsymbol{\Lambda})\boldsymbol{S}^{-1} \\ &= \boldsymbol{S}\boldsymbol{\Lambda}_j\boldsymbol{S}^{-1} \end{aligned}$$

where $\boldsymbol{\Lambda}_j$ is a diagonal matrix with jth diagonal element equal to 0 for $j = 1, 2, \cdots, n$. Therefore, we can obtain

$$\begin{aligned} & (\lambda_1 \boldsymbol{I} - \boldsymbol{A})(\lambda_2 \boldsymbol{I} - \boldsymbol{A}) \cdots (\lambda_n \boldsymbol{I} - \boldsymbol{A}) \\ =\ & \boldsymbol{S}\boldsymbol{\Lambda}_1\boldsymbol{S}^{-1}\boldsymbol{S}\boldsymbol{\Lambda}_2\boldsymbol{S}^{-1} \cdots \boldsymbol{S}\boldsymbol{\Lambda}_n\boldsymbol{S}^{-1} \\ =\ & \boldsymbol{S}\boldsymbol{\Lambda}_1\boldsymbol{\Lambda}_2 \cdots \boldsymbol{\Lambda}_n\boldsymbol{S}^{-1} \\ =\ & \boldsymbol{S}\boldsymbol{O}\boldsymbol{S}^{-1} \\ =\ & \boldsymbol{O} \end{aligned}$$

where $\boldsymbol{O}$ is the zero matrix.

4. (a) Let $\boldsymbol{u}_k = \begin{bmatrix} M_{k+1} \\ M_k \end{bmatrix}$, and we can have

$$\boldsymbol{u}_{k+1} = \begin{bmatrix} M_{k+2} \\ M_{k+1} \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} M_{k+1} \\ M_k \end{bmatrix} = \boldsymbol{A}\boldsymbol{u}_k$$

with

$$\boldsymbol{u}_0 = \begin{bmatrix} M_1 \\ M_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then we obtain $\boldsymbol{u}_k = \boldsymbol{A}\boldsymbol{u}_{k-1} = \boldsymbol{A}^2\boldsymbol{u}_{k-2} = \boldsymbol{A}^k\boldsymbol{u}_0$. To find $\boldsymbol{A}^k$, we first find the eigenvalues of $\boldsymbol{A}$. Consider

$$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = (\lambda+1)(\lambda+2) = 0.$$

Thus, we have $\lambda = -1, -2$. Then we need to find the corresponding eigenvectors:

$$\boldsymbol{A} - \lambda_1\boldsymbol{I} = \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Longrightarrow \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$\boldsymbol{A} - \lambda_2\boldsymbol{I} = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \Longrightarrow \boldsymbol{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Therefore, we obtain

$$\boldsymbol{A} = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}^{-1}.$$

Finally, we can have

$$\begin{aligned} \boldsymbol{u}_k &= \boldsymbol{A}^k\boldsymbol{u}_0 = \boldsymbol{S}\boldsymbol{\Lambda}^k\boldsymbol{S}^{-1}\boldsymbol{u}_0 \\ &= \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^k & 0 \\ 0 & (-2)^k \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^k & 0 \\ 0 & (-2)^k \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^k & 0 \\ 0 & (-2)^k \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} M_{k+1} \\ M_k \end{bmatrix}. \end{aligned}$$

As a result, $M_k = (-1)^k - (-2)^k$.

(b) Let $\boldsymbol{u} = \begin{bmatrix} u' \\ u \end{bmatrix}$, and we can obtain

$$\boldsymbol{u}' = \begin{bmatrix} u'' \\ u' \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u' \\ u \end{bmatrix} = \boldsymbol{A}\boldsymbol{u}$$

with

$$\boldsymbol{u}(0) = \begin{bmatrix} u'(0) \\ u(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Therefore, we have

$$\begin{aligned} \boldsymbol{u}(t) &= e^{\boldsymbol{A}t}\boldsymbol{u}(0) = \boldsymbol{S}e^{\boldsymbol{\Lambda}t}\boldsymbol{S}^{-1}\boldsymbol{u}(0) \\ &= \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} u'(t) \\ u(t) \end{bmatrix}. \end{aligned}$$

As a result, $u(t) = e^{-t} - e^{-2t}$.

5. (a) Since $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$ is a scalar and $\boldsymbol{A}$ is real skew-symmetric, we have

$$\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = (\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x})^T = \boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{x} = -\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}.$$

Therefore, $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = 0$ for every real vector $\boldsymbol{x}$.

(b) Suppose $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$. Then we can take complex conjugation on both sides and obtain

$$\overline{\boldsymbol{A}\boldsymbol{x}} = \overline{\lambda\boldsymbol{x}} \implies \overline{\boldsymbol{A}}\overline{\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}}.$$

Since $\boldsymbol{A}$ is real, we have $\overline{\boldsymbol{A}} = \boldsymbol{A}$. Then we can have

$$\begin{aligned} & \boldsymbol{A}\overline{\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}} \\ \implies & \overline{\boldsymbol{x}}^T\boldsymbol{A}^T = \overline{\lambda}\overline{\boldsymbol{x}}^T \\ \implies & \overline{\boldsymbol{x}}^T\boldsymbol{A} = -\overline{\lambda}\overline{\boldsymbol{x}}^T \end{aligned}$$

where the last equality follows since $\boldsymbol{A}$ is skew-symmetric. Consider $\overline{\boldsymbol{x}}^T\boldsymbol{A}\boldsymbol{x}$, and we can have

$$\overline{\boldsymbol{x}}^T(\boldsymbol{A}\boldsymbol{x}) = \overline{\boldsymbol{x}}^T(\lambda\boldsymbol{x}) = \lambda\overline{\boldsymbol{x}}^T\boldsymbol{x} = \lambda\|\boldsymbol{x}\|^2$$

along with

$$\left(\overline{\boldsymbol{x}}^T\boldsymbol{A}\right)\boldsymbol{x} = \left(-\overline{\lambda}\overline{\boldsymbol{x}}^T\right)\boldsymbol{x} = -\overline{\lambda}\left(\overline{\boldsymbol{x}}^T\boldsymbol{x}\right) = -\overline{\lambda}\|\boldsymbol{x}\|^2.$$

Hence, we can have

$$\lambda = -\overline{\lambda}.$$

Therefore, a real skew-symmetric matrix has pure imaginary eigenvalues.

(c) Since $\boldsymbol{A}$ is a real matrix, its eigenvalues come in conjugate pairs. From (b), all eigenvalues of $\boldsymbol{A}$ are pure imaginary. Since the determinant of $\boldsymbol{A}$ is the product of all eigenvalues and $ic(-ic) = c^2 \geq 0$ for $c \geq 0$, the determinant of $\boldsymbol{A}$ is positive or zero.

6. (a) We find the eigenvalues of the matrix $\boldsymbol{A}$ first. Consider

$$\begin{aligned} \det(\boldsymbol{A} - \lambda\boldsymbol{I}) &= \begin{vmatrix} 10-\lambda & -6 \\ -6 & 10-\lambda \end{vmatrix} \\ &= (10-\lambda)^2 - 36 \\ &= (4-\lambda)(16-\lambda) = 0. \end{aligned}$$

We can obtain $\lambda = 4, 16$. For $\lambda_1 = 4$, we have

$$\boldsymbol{A} - \lambda_1\boldsymbol{I} = \begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix} \implies \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

and the corresponding unit eigenvector is

$$\boldsymbol{x}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = 16$, we have

$$\boldsymbol{A} - \lambda_2 \boldsymbol{I} = \begin{bmatrix} -6 & -6 \\ -6 & -6 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

and the corresponding unit eigenvector is

$$\boldsymbol{x}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Therefore, we can obtain an orthogonal matrix

$$\boldsymbol{Q} = [\boldsymbol{x}_1, \boldsymbol{x}_2] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and a diagonal matrix

$$\boldsymbol{\Lambda} = \begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix}$$

such that $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T$.

(b) Perform elimination, and we can have

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T \\ &= \begin{bmatrix} 1 & 0 \\ -3/5 & 1 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 32/5 \end{bmatrix} \begin{bmatrix} 1 & -3/5 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

We then have $\boldsymbol{A} = \boldsymbol{C}\boldsymbol{C}^T$ where

$$\begin{aligned} \boldsymbol{C} &= \boldsymbol{L}\sqrt{\boldsymbol{D}} \\ &= \begin{bmatrix} 1 & 0 \\ -3/5 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{32/5} \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{10} & 0 \\ -\sqrt{\frac{18}{5}} & \sqrt{\frac{32}{5}} \end{bmatrix}. \end{aligned}$$

7. (a) Let $\boldsymbol{x}_1 = (1, 1, 1)^T$, and we have $\boldsymbol{x}_1^T \boldsymbol{A} \boldsymbol{x}_1 = 33 > 0$. Let $\boldsymbol{x}_2 = (1, -0.1, -0.2)^T$, and we have $\boldsymbol{x}_2^T \boldsymbol{A} \boldsymbol{x}_2 = -0.03 < 0$. Therefore, $\boldsymbol{A}$ is indefinite.

(b) Performing elimination on $\boldsymbol{B}$ without row exchanges, we can have

$$\boldsymbol{R_B} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 5/3 \end{bmatrix}.$$

Since $\boldsymbol{B}$ is a real symmetric matrix and all pivots (without row exchanges) are positive, $\boldsymbol{B}$ is positive definite.

(c) $\boldsymbol{C} = -\boldsymbol{B}$ is also a symmetric matrix. Since $\boldsymbol{B}$ is positive definite, we have

$$\boldsymbol{x}^T\boldsymbol{C}\boldsymbol{x} = -\boldsymbol{x}^T\boldsymbol{B}\boldsymbol{x} < 0$$

for every nonzero vector $\boldsymbol{x}$. Therefore, $\boldsymbol{C}$ is negative definite.

(d) We know that $\boldsymbol{D} = \boldsymbol{A}^{-1}$ is also a symmetric matrix. For every nonzero vector $\boldsymbol{x}$, we have

$$\boldsymbol{x}^T\boldsymbol{D}\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{A}^{-1}\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{A}^{-1}\boldsymbol{x} = (\boldsymbol{A}^{-1}\boldsymbol{x})^T\boldsymbol{A}(\boldsymbol{A}^{-1}\boldsymbol{x}) = \boldsymbol{x}'^T\boldsymbol{A}\boldsymbol{x}'$$

for some nonzero vector $\boldsymbol{x}'$. Since $\boldsymbol{A}$ is indefinite, $\boldsymbol{D}$ is also indefinite.

8. (i) Consider

$$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = -(\lambda-2)^2(\lambda-1) = 0.$$

Thus, we have $\lambda = 2, 2, 1$. For $\lambda_1 = 2$, the AM of λ_1 equals 2. Besides,

$$\boldsymbol{A} - \lambda_1\boldsymbol{I} = \begin{bmatrix} -5 & 3 & -2 \\ -7 & 4 & -3 \\ 1 & -1 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

and the GM of λ_1 equals 1. For $\lambda_2 = 1$, the AM of λ_2 equals 1. Besides,

$$\boldsymbol{A} - \lambda_2\boldsymbol{I} = \begin{bmatrix} -4 & 3 & -2 \\ -7 & 5 & -3 \\ 1 & -1 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix}$$

and the GM of λ_2 equals 1. Therefore, the Jordan form of $\boldsymbol{A}$ is

$$\boldsymbol{J}_A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(ii) Consider

$$\det(\boldsymbol{B} - \lambda\boldsymbol{I}) = (\lambda-2)^2(\lambda-1) = 0.$$

Thus, we have $\lambda = 2, 2, 1$. For $\lambda_1 = 2$, the AM of λ_1 equals 2. Besides,

$$\boldsymbol{B} - \lambda_1\boldsymbol{I} = \begin{bmatrix} -2 & -1 & -1 \\ -3 & -3 & -2 \\ 7 & 5 & 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

and the GM of λ_1 equals 1. For $\lambda_2 = 1$, the AM of λ_2 equals 1. Besides,

$$\boldsymbol{B} - \lambda_2\boldsymbol{I} = \begin{bmatrix} -1 & -1 & -1 \\ -3 & -2 & -2 \\ 7 & 5 & 5 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and the GM of λ_1 equals 1. Therefore, the Jordan form of $\boldsymbol{B}$ is

$$\boldsymbol{J}_B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $\boldsymbol{J}_A = \boldsymbol{J}_B$ from (i) and (ii), $\boldsymbol{A}$ is similar to $\boldsymbol{B}$.