EE 2030 Linear Algebra Chi-chao Chao
Spring 2012

Homework Assignment No. 6
Due 10:10am, June 13, 2012

`Reading:` Strang, Section 6.7, Chapter 7.

`Problems for Solution:`

1. Suppose the singular value decomposition $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$ has

$$\boldsymbol{U} = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{V} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

   (a) Find the eigenvalues of $\boldsymbol{A}^T\boldsymbol{A}$.

   (b) Find a basis for the nullspace of $\boldsymbol{A}$.

   (c) Find a basis for the column space of $\boldsymbol{A}$.

   (d) Find a singular value decomposition of $-\boldsymbol{A}^T$.

2. Suppose $\boldsymbol{A}$ is a 2 by 2 symmetric matrix with unit eigenvectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. If its eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$, what are the matrices $\boldsymbol{U}$, $\boldsymbol{\Sigma}$, and $\boldsymbol{V}$ in its singular value decomposition?

3. Is each of the following transformations linear? If yes, prove it; otherwise, find a counterexample.

   (a) $T(v_1, v_2) = (v_1, v_1)$.

   (b) $T(v_1, v_2) = (0, 1)$.

   (c) $T(v_1, v_2) = v_1 v_2$.

   (d) $T(v_1, v_2) = (v_1, v_2)$ except that $T(0, v_2) = (0, 0)$.

4. A linear transformation T from V to W has an inverse from W to V when the range is all of W and the kernel contains only $\boldsymbol{v} = \boldsymbol{0}$. Then $T(\boldsymbol{v}) = \boldsymbol{w}$ has one solution $\boldsymbol{v}$ for each $\boldsymbol{w}$ in W. We thus say that T is invertible and denote its inverse by T^{-1}.

   (a) Suppose $T(\boldsymbol{v}_1) = \boldsymbol{w}_1 + \boldsymbol{w}_2 + \boldsymbol{w}_3$, $T(\boldsymbol{v}_2) = \boldsymbol{w}_2 + \boldsymbol{w}_3$, and $T(\boldsymbol{v}_3) = \boldsymbol{w}_3$ where $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ and $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$ are bases for V and W, respectively. Find the matrix $\boldsymbol{A}$ for T using these bases.

   (b) Invert the matrix $\boldsymbol{A}$ in (a). What are $T^{-1}(\boldsymbol{w}_1)$, $T^{-1}(\boldsymbol{w}_2)$, and $T^{-1}(\boldsymbol{w}_3)$?

5. Define the linear operator T on $\mathcal{R}^3$ by

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} v_1 + 2v_3 \\ -v_2 - 2v_3 \\ 2v_1 - 2v_2 \end{bmatrix}.$$

Find a basis for $\mathcal{R}^3$ such that the matrix representation for T in this basis is a diagonal matrix.

6. Let P_2 be the vector space of all polynomials of degree at most 2, i.e., $P_2 = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in \mathcal{R}\}$. Consider the linear operator L on P_2 defined by

$$L(p(x)) = xp'(x) + p''(x)$$

where $p'(x)$ is the derivative of $p(x)$ and $p''(x)$ is the second derivative of $p(x)$.

(a) Find the matrix $\boldsymbol{A}$ representing L with respect to the basis $\{1, x, x^2\}$.

(b) Find the matrix $\boldsymbol{B}$ representing L with respect to the basis $\{1, x, 1 + x^2\}$.

(c) Find the matrix $\boldsymbol{M}$ such that $\boldsymbol{B} = \boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}$.

7. Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}.$$

(a) Find the polar decomposition $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{H}$, where $\boldsymbol{Q}$ is an orthogonal matrix and $\boldsymbol{H}$ is a positive semidefinite matrix.

(b) Find the pseudoinverse $\boldsymbol{A}^+$ of $\boldsymbol{A}$.

8. Find the shortest least squares solution to

$$\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \boldsymbol{b}.$$