Solution to Homework Assignment No. 6

1. We have $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$ is a 3 by 4 matrix where

$$\boldsymbol{U} = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}, \ \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \boldsymbol{V} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

(a) Since $\boldsymbol{A}^T\boldsymbol{A}$ is 4 by 4, it has 4 eigenvalues. The nonzero eigenvalues of $\boldsymbol{A}^T\boldsymbol{A}$ are the squares of the singular values of $\boldsymbol{A}$, which are given by $1^2 = 1$ and $4^2 = 16$. Therefore, the eigenvalues of $\boldsymbol{A}^T\boldsymbol{A}$ are 1, 16, 0, 0.

(b) Since there are 2 nonzero singular values of $\boldsymbol{A}$, the rank of $\boldsymbol{A}$ is 2. Therefore, the dimension of the nullspace of $\boldsymbol{A} = 4 - 2 = 2$. A basis for the nullspace of $\boldsymbol{A}$ can be obtained as the last two columns of $\boldsymbol{V}$, i.e.,

$$\begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}.$$

(c) Since the dimension of the column space of $\boldsymbol{A}$ is 2, a basis for the column space of $\boldsymbol{A}$ can be obtained as the first two columns of $\boldsymbol{U}$, i.e.,

$$\begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}.$$

(d) A singular value decomposition of $-\boldsymbol{A}^T$ can be given by

$$-\boldsymbol{A}^T = -(\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T)^T = -\boldsymbol{V}\boldsymbol{\Sigma}^T\boldsymbol{U}^T = \boldsymbol{U}'\boldsymbol{\Sigma}'\boldsymbol{V}'^T$$

where $\boldsymbol{U}' = -\boldsymbol{V}$, $\boldsymbol{\Sigma}' = \boldsymbol{\Sigma}^T$ and $\boldsymbol{V}' = \boldsymbol{U}$.

2. Since $\boldsymbol{A}$ is a symmetry matrix, we have $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{u}_1 = \boldsymbol{A}\boldsymbol{A}\boldsymbol{u}_1 = \lambda_1\boldsymbol{A}\boldsymbol{u}_1 = \lambda_1^2\boldsymbol{u}_1$ and $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{u}_2 = \boldsymbol{A}\boldsymbol{A}\boldsymbol{u}_2 = \lambda_2\boldsymbol{A}\boldsymbol{u}_2 = \lambda_2^2\boldsymbol{u}_2$. Therefore, this implies that $\sigma_1^2 = \lambda_1^2 = 9$ and $\sigma_2^2 = \lambda_2^2 = 4$. We have $\sigma_1 = \lambda_1 = 3$ and $\sigma_2 = -\lambda_2 = 2$. Besides, $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are the unit eigenvectors of $\boldsymbol{A}^T\boldsymbol{A}$. Furthermore, we can know that $\boldsymbol{A}\boldsymbol{u}_1 = \lambda_1\boldsymbol{u}_1 = \sigma_1\boldsymbol{u}_1$ and $\boldsymbol{A}\boldsymbol{u}_2 = \lambda_2\boldsymbol{u}_2 = \sigma_2(-\boldsymbol{u}_2)$. As a result, the matrices are

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{u}_1 & -\boldsymbol{u}_2 \end{bmatrix}, \ \boldsymbol{\Sigma} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \text{ and } \boldsymbol{V} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix}.$$

3. (a) True. Let $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$. We can check the following two conditions:

- $T(\boldsymbol{v}+\boldsymbol{w}) = T(v_1+w_1, v_2+w_2) = (v_1+w_1, v_1+w_1) = (v_1, v_1)+(w_1, w_1) = T(\boldsymbol{v}) + T(\boldsymbol{w})$.
- $T(c\boldsymbol{v}) = T(cv_1, cv_2) = (cv_1, cv_1) = c(v_1, v_1) = cT(\boldsymbol{v})$ for all c.

Therefore, it is linear.

(b) False. Let $\boldsymbol{v} = (1, 1)$. Since

$$T(0 \cdot \boldsymbol{v}) = T(0, 0) = (0, 1) \neq (0, 0) = 0 \cdot (0, 1) = 0 \cdot T(\boldsymbol{v})$$

it is not linear.

(c) False. Let $\boldsymbol{v} = (1, 1)$ and $\boldsymbol{w} = (2, 2)$. Since

$$T(\boldsymbol{v} + \boldsymbol{w}) = T(3, 3) = 9 \neq 5 = T(1, 1) + T(2, 2) = T(\boldsymbol{v}) + T(\boldsymbol{w})$$

it is not linear.

(d) False. Let $\boldsymbol{v} = (1, 1)$ and $\boldsymbol{w} = (0, 2)$. Since

$$T(\boldsymbol{v} + \boldsymbol{w}) = T(1, 3) = (1, 3) \neq (1, 1) = T(1, 1) + T(0, 2) = T(\boldsymbol{v}) + T(\boldsymbol{w})$$

it is not linear.

4. (a) Since $T(\boldsymbol{v}_1) = 1 \cdot \boldsymbol{w}_1 + 1 \cdot \boldsymbol{w}_2 + 1 \cdot \boldsymbol{w}_3$, $T(\boldsymbol{v}_2) = 0 \cdot \boldsymbol{w}_1 + 1 \cdot \boldsymbol{w}_2 + 1 \cdot \boldsymbol{w}_3$, and $T(\boldsymbol{v}_3) = 0 \cdot \boldsymbol{w}_1 + 0 \cdot \boldsymbol{w}_2 + 1 \cdot \boldsymbol{w}_3$, we have the matrix $\boldsymbol{A}$ for T as

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

(b) We have

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Therefore, we can obtain $T^{-1}(\boldsymbol{w}_1) = \boldsymbol{v}_1 - \boldsymbol{v}_2$, $T^{-1}(\boldsymbol{w}_2) = \boldsymbol{v}_2 - \boldsymbol{v}_3$, and $T^{-1}(\boldsymbol{w}_3) = \boldsymbol{v}_3$.

5. For the standard basis, the matrix which represents this T is

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}.$$

The eigenvectors for this matrix are $(-1, 2, 2)$, $(-2, -2, 1)$, and $(2, -1, 2)$. Therefore, we can find the basis $\{(-1, 2, 2), (-2, -2, 1), (2, -1, 2)\}$ such that the matrix representation for T in this basis is a diagonal matrix.

6. (a) Let $\beta = \{1, x, x^2\}$. Since

$$\begin{aligned} L(1) &= 0 &&= 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\ L(x) &= x &&= 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2 \\ L(x^2) &= 2x^2 + 2 &&= 2 \cdot 1 + 0 \cdot x + 2 \cdot x^2 \end{aligned}$$

we have

$$\boldsymbol{A} = [L]_\beta = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(b) Let $\gamma = \{1, x, 1+x^2\}$. Since

$$\begin{array}{rcl} L(1) &=& 0 &=& 0\cdot 1 + 0\cdot x + 0\cdot(1+x^2) \\ L(x) &=& x &=& 0\cdot 1 + 1\cdot x + 0\cdot(1+x^2) \\ L(1+x^2) &=& 2x^2+2 &=& 0\cdot 1 + 0\cdot x + 2\cdot(1+x^2) \end{array}$$

we have

$$\boldsymbol{B} = [L]_\gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(c) Let I be the identity transformation, and we have $\boldsymbol{M} = [I]_\gamma^\beta$. Since

$$\begin{array}{rcl} I(1) &=& 1 &=& 1\cdot 1 + 0\cdot x + 0\cdot x^2 \\ I(x) &=& x &=& 0\cdot 1 + 1\cdot x + 0\cdot x^2 \\ I(1+x^2) &=& 1+x^2 &=& 1\cdot 1 + 0\cdot x + 1\cdot x^2 \end{array}$$

we can obtain

$$\boldsymbol{M} = [I]_\gamma^\beta = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

7. Perform the singular value decomposition, and we can have

$$\boldsymbol{A} = \boldsymbol{U\Sigma V}^T = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$$

(a)

$$\boldsymbol{A} = \boldsymbol{QH} = (\boldsymbol{UV}^T)(\boldsymbol{V\Sigma V}^T) = \frac{1}{\sqrt{50}}\begin{bmatrix} 7 & -1 \\ 1 & 7 \end{bmatrix}\frac{1}{\sqrt{50}}\begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}.$$

(b)

$$\begin{array}{rcl} \boldsymbol{A}^+ &=& \boldsymbol{V\Sigma}^+\boldsymbol{U}^T \\ &=& \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{50} & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \\ &=& \frac{1}{50}\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}. \end{array}$$

8. Perform the singular value decomposition, and we can have

$$\boldsymbol{A} = \boldsymbol{U\Sigma V}^T = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & \sqrt{2} & \sqrt{3} \\ 1 & \sqrt{2} & -\sqrt{3} \\ 2 & -\sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\frac{1}{\sqrt{6}}\begin{bmatrix} 2 & 1 & 1 \\ \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 0 & \sqrt{3} & -\sqrt{3} \end{bmatrix}.$$

Then we can obtain

$$
\begin{aligned}
\boldsymbol{A}^+ &= \boldsymbol{V}\boldsymbol{\Sigma}^+\boldsymbol{U}^T \\
&= \frac{1}{\sqrt{6}}\begin{bmatrix} 2 & \sqrt{2} & 0 \\ 1 & -\sqrt{2} & \sqrt{3} \\ 1 & -\sqrt{2} & -\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 & 2 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ \sqrt{3} & -\sqrt{3} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1/2 & 1/2 & 0 \\ -1/4 & -1/4 & 1/2 \\ -1/4 & -1/4 & 1/2 \end{bmatrix}.
\end{aligned}
$$

Therefore, the shortest least squares solution is

$$
\boldsymbol{A}^+\boldsymbol{b} = \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \end{bmatrix}.
$$