EE 2030 Linear Algebra Chi-chao Chao
Spring 2013

Homework Assignment No. 1
Due 10:10am, March 8, 2013

`Reading:` Strang, Chapters 1 and 2.

`Problems for Solution:`

1. (a) Find the pivot and solution for the following system of linear equations:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 9 \end{bmatrix}.$$

   (b) Solve

$$\boldsymbol{LUx} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

   without multiplying $\boldsymbol{LU}$ to find $\boldsymbol{A}$.

2. Suppose $\boldsymbol{A}$ is an invertible matrix and let $\boldsymbol{A}^m = \underbrace{\boldsymbol{AA}\cdots\boldsymbol{A}}_{m}$.

   (a) Show that $\boldsymbol{A}^2$ is invertible and $(\boldsymbol{A}^2)^{-1} = (\boldsymbol{A}^{-1})^2$.

   (b) Use mathematical induction to show that $\boldsymbol{A}^m$ is invertible and $(\boldsymbol{A}^m)^{-1} = (\boldsymbol{A}^{-1})^m$, for $m = 1, 2, 3, \ldots$.

3. (a) Do Problem 40 of Problem Set 2.5 in p. 92 of Strang. Specifically,

$$\boldsymbol{A} = \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

   (b) Guess the result of $\boldsymbol{A}^{-1}$ if

$$\boldsymbol{A} = \begin{bmatrix} 1 & -a & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 \\ 0 & 0 & 1 & -c & 0 \\ 0 & 0 & 0 & 1 & -d \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

   Then multiply to confirm.

4. True or false. (If it is true, prove it. Otherwise, find a counterexample.)

   (a) Every n by n matrix with 1's down the main diagonal is invertible.

   (b) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric matrices, then $\boldsymbol{AB}+\boldsymbol{BA}$ is also symmetric.

5. Let $\boldsymbol{A}$ be an n by n matrix.

   (a) Suppose there exists an n by n matrix $\boldsymbol{B}$ such that $\boldsymbol{AB} = \boldsymbol{I}$. Show that $\boldsymbol{A}$ is invertible and $\boldsymbol{B} = \boldsymbol{A}^{-1}$. (*Hint:* Show that $\boldsymbol{A}$ is nonsingular. Suppose $\boldsymbol{A}$ is singular. Then elimination will lead to a zero row. Hence show by contradiction that it is impossible to have $\boldsymbol{AB} = \boldsymbol{I}$.)

   (b) Suppose there exists an n by n matrix $\boldsymbol{C}$ such that $\boldsymbol{CA} = \boldsymbol{I}$. Show that $\boldsymbol{A}$ is invertible and $\boldsymbol{C} = \boldsymbol{A}^{-1}$. (*Hint:* Consider $\boldsymbol{A}^T\boldsymbol{C}^T = \boldsymbol{I}^T = \boldsymbol{I}$. Then use (a).)

6. Factor the following symmetric matrices into $\boldsymbol{LDL}^T$:

$$\boldsymbol{P} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \quad \text{and} \quad \boldsymbol{T} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

   where $\boldsymbol{P}$ is a *symmetric Pascal* matrix mentioned in Worked Example 2.6A in p. 101 of Strang and $\boldsymbol{T}$ is a *tridiagonal* matrix in Problem 20 of Problem Set 2.6 in p. 105 of Strang.

7. If $\boldsymbol{A} = \boldsymbol{L}_1\boldsymbol{D}_1\boldsymbol{U}_1$ and $\boldsymbol{A} = \boldsymbol{L}_2\boldsymbol{D}_2\boldsymbol{U}_2$, where the $\boldsymbol{L}$'s are lower triangular with unit diagonal, the $\boldsymbol{U}$'s are upper triangular with unit diagonal, and the $\boldsymbol{D}$'s are diagonal matrices with no zeros on the diagonal, prove that $\boldsymbol{L}_1 = \boldsymbol{L}_2$, $\boldsymbol{D}_1 = \boldsymbol{D}_2$, and $\boldsymbol{U}_1 = \boldsymbol{U}_2$. Note that the proof can be decomposed into the following two steps:

   (a) Derive the equation $\boldsymbol{L}_1^{-1}\boldsymbol{L}_2\boldsymbol{D}_2 = \boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1}$ and explain why one side is lower triangular and the other side is upper triangular.

   (b) Compare the main diagonals in the equation in (a), and then compare the off-diagonals.

   In your proof, you may use the following assertions without proving them:

   (i) A lower (upper) triangular matrix with unit diagonal is invertible and its inverse is still lower (upper) triangular with unit diagonal.

   (ii) The product of two lower (upper) triangular matrices with unit diagonal is still lower (upper) triangular with unit diagonal.

   (iii) The product of a lower (upper) triangular matrix and a diagonal matrix is lower (upper) triangular.

8. Factor the following matrix into $\boldsymbol{PA} = \boldsymbol{LU}$. Also factor it into $\boldsymbol{A} = \boldsymbol{L}_1\boldsymbol{P}_1\boldsymbol{U}_1$.

$$\boldsymbol{A} = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 2 & 2 \\ 2 & 6 & 7 \end{bmatrix}.$$