Solution to Homework Assignment No. 1

1. (a) We can perform Gaussian elimination as follows:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & 3 & 4 & 4 \\ 3 & 5 & 8 & 9 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & -2 & -2 \\ 0 & -1 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & -2 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

Hence the pivots are 1, -1 and 1, and by back substitution the solution is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.$$

(b) Let $\boldsymbol{Ux} = \boldsymbol{c}$ and $\boldsymbol{Lc} = \boldsymbol{b}$. First, we solve $\boldsymbol{c}$ from $\boldsymbol{Lc} = \boldsymbol{b}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}.$$

Next, we solve $\boldsymbol{x}$ from $\boldsymbol{Ux} = \boldsymbol{c}$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

2. (a) If $\boldsymbol{A}$ is invertible, we have $\boldsymbol{AA}^{-1} = \boldsymbol{I}$ and $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$. Since we have $\boldsymbol{A}^2 = \boldsymbol{AA}$ and $(\boldsymbol{A}^{-1})^2 = \boldsymbol{A}^{-1}\boldsymbol{A}^{-1}$, we can have

$$\boldsymbol{A}^2 \left(\boldsymbol{A}^{-1}\right)^2 = \boldsymbol{A}\left(\boldsymbol{AA}^{-1}\right)\boldsymbol{A}^{-1} = \boldsymbol{AIA}^{-1} = \boldsymbol{AA}^{-1} = \boldsymbol{I}$$
$$\left(\boldsymbol{A}^{-1}\right)^2 \boldsymbol{A}^2 = \boldsymbol{A}^{-1}\left(\boldsymbol{A}^{-1}\boldsymbol{A}\right)\boldsymbol{A} = \boldsymbol{A}^{-1}\boldsymbol{IA} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}.$$

Hence $\boldsymbol{A}^2$ is invertible and $(\boldsymbol{A}^2)^{-1} = (\boldsymbol{A}^{-1})^2$.

(b) For $m = 1$, the statement is obviously true. Assume $m = k$ is true. Then considering $m = k + 1$, we can have

$$\boldsymbol{A}^{k+1}\left(\boldsymbol{A}^{-1}\right)^{k+1} = \boldsymbol{A}^{k}\left(\boldsymbol{A}\boldsymbol{A}^{-1}\right)\left(\boldsymbol{A}^{-1}\right)^{k} = \boldsymbol{A}^{k}\boldsymbol{I}\left(\boldsymbol{A}^{-1}\right)^{k} = \boldsymbol{A}^{k}\left(\boldsymbol{A}^{-1}\right)^{k} = \boldsymbol{I}$$
$$\left(\boldsymbol{A}^{-1}\right)^{k+1}\boldsymbol{A}^{k+1} = \left(\boldsymbol{A}^{-1}\right)^{k}\left(\boldsymbol{A}^{-1}\boldsymbol{A}\right)\boldsymbol{A}^{k} = \left(\boldsymbol{A}^{-1}\right)^{k}\boldsymbol{I}\boldsymbol{A}^{k} = \left(\boldsymbol{A}^{-1}\right)^{k}\boldsymbol{A}^{k} = \boldsymbol{I}.$$

Hence the statement holds for $m = k + 1$. By induction, this statement is true.

3. (a) By applying Gaussian-Jordan elimination, we have

$$\left[\begin{array}{cccc|cccc} 1 & -a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -c & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & a & ab & abc \\ 0 & 1 & 0 & 0 & 0 & 1 & b & bc \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right].$$

Hence

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 1 & a & ab & abc \\ 0 & 1 & b & bc \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) From (a), we can guess

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 1 & a & ab & abc & abcd \\ 0 & 1 & b & bc & bcd \\ 0 & 0 & 1 & c & cd \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This can be confirmed by multiplying them:

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \begin{bmatrix} 1 & -a & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 \\ 0 & 0 & 1 & -c & 0 \\ 0 & 0 & 0 & 1 & -d \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & ab & abc & abcd \\ 0 & 1 & b & bc & bcd \\ 0 & 0 & 1 & c & cd \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \boldsymbol{I}$$
$$\boldsymbol{A}^{-1}\boldsymbol{A} = \begin{bmatrix} 1 & a & ab & abc & abcd \\ 0 & 1 & b & bc & bcd \\ 0 & 0 & 1 & c & cd \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 \\ 0 & 0 & 1 & -c & 0 \\ 0 & 0 & 0 & 1 & -d \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \boldsymbol{I}.$$

4. (a) False. A counterexample is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Since it does not have a full set of pivots, it is not invertible.

(b) True. One can check that

$$(\boldsymbol{AB}+\boldsymbol{BA})^T = (\boldsymbol{AB})^T + (\boldsymbol{BA})^T = \boldsymbol{B}^T\boldsymbol{A}^T + \boldsymbol{A}^T\boldsymbol{B}^T = \boldsymbol{BA} + \boldsymbol{AB} = \boldsymbol{AB} + \boldsymbol{BA}.$$

5. (a) Suppose $\boldsymbol{A}$ is singular. By elimination we can assume that there is an invertible matrix $\boldsymbol{M}$ such that a row of $\boldsymbol{MA}$ is zero. Since $\boldsymbol{MAB} = \boldsymbol{MI} = \boldsymbol{M}$, a row of $\boldsymbol{M}$ is zero, which reaches a contradiction because $\boldsymbol{M}$ is invertible. Hence $\boldsymbol{A}$ is nonsingular and thus invertible. We can therefore obtain $\boldsymbol{B} = (\boldsymbol{A}^{-1}\boldsymbol{A})\boldsymbol{B} = \boldsymbol{A}^{-1}(\boldsymbol{AB}) = \boldsymbol{A}^{-1}\boldsymbol{I} = \boldsymbol{A}^{-1}$.

(b) Consider $\boldsymbol{A}^T\boldsymbol{C}^T = \boldsymbol{I}^T = \boldsymbol{I}$. From (a), we have that $\boldsymbol{A}^T$ is invertible and $(\boldsymbol{A}^T)^{-1} = \boldsymbol{C}^T$. Since $(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^T = \boldsymbol{C}^T$, we can obtain $\boldsymbol{A}^{-1} = \boldsymbol{C}$.

6. (a) Performing elimination, we can have

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \overset{\boldsymbol{E}_{21}}{\Longrightarrow} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 6 \end{bmatrix} \overset{\boldsymbol{E}_{31}}{\Longrightarrow} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix} \overset{\boldsymbol{E}_{32}}{\Longrightarrow} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{U}.$$

This procedure can be viewed as

$$\boldsymbol{E}_{32}\boldsymbol{E}_{31}\boldsymbol{E}_{21}\boldsymbol{A} = \boldsymbol{U}$$

where

$$\boldsymbol{E}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \boldsymbol{E}_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \text{ and } \boldsymbol{E}_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}.$$

Recording the elimination steps and changing the signs of the off-diagonal elements, we can have

$$\boldsymbol{L} = \boldsymbol{E}_{21}^{-1}\boldsymbol{E}_{31}^{-1}\boldsymbol{E}_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}.$$

We also find that $\boldsymbol{U} = \boldsymbol{DL}^T$ where

$$\boldsymbol{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We can therefore obtain $\boldsymbol{A} = \boldsymbol{LDL}^T$ as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Performing elimination, we can have

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix} \overset{\boldsymbol{E}_{21}}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix} \overset{\boldsymbol{E}_{32}}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix} \overset{\boldsymbol{E}_{43}}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \boldsymbol{U}.$$

This procedure can be viewed as

$$\boldsymbol{E}_{43}\boldsymbol{E}_{32}\boldsymbol{E}_{21}\boldsymbol{A} = \boldsymbol{U}$$

where

$$\boldsymbol{E}_{21}=\begin{bmatrix}1&0&0&0\\-2&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\ \boldsymbol{E}_{32}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&0&0&1\end{bmatrix},\ \text{and } \boldsymbol{E}_{43}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-1&1\end{bmatrix}.$$

Recording the elimination steps and changing the signs of the off-diagonal elements, we can have

$$\boldsymbol{L}=\boldsymbol{E}_{21}^{-1}\boldsymbol{E}_{32}^{-1}\boldsymbol{E}_{43}^{-1}=\begin{bmatrix}1&0&0&0\\2&1&0&0\\0&-1&1&0\\0&0&1&1\end{bmatrix}.$$

We also find that $\boldsymbol{U}=\boldsymbol{D}\boldsymbol{L}^T$ where

$$\boldsymbol{D}=\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&3&0\\0&0&0&1\end{bmatrix}.$$

We can therefore obtain $\boldsymbol{A}=\boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T$ as

$$\begin{bmatrix}1&2&0&0\\2&3&1&0\\0&1&2&3\\0&0&3&4\end{bmatrix}=\begin{bmatrix}1&0&0&0\\2&1&0&0\\0&-1&1&0\\0&0&1&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&3&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&2&0&0\\0&1&-1&0\\0&0&1&1\\0&0&0&1\end{bmatrix}.$$

7. (a) By (i), $\boldsymbol{L}_1^{-1}$ and $\boldsymbol{U}_2^{-1}$ both exist. Given $\boldsymbol{A}=\boldsymbol{L}_1\boldsymbol{D}_1\boldsymbol{U}_1$ and $\boldsymbol{A}=\boldsymbol{L}_2\boldsymbol{D}_2\boldsymbol{U}_2$, we can have

$$\begin{aligned}&\boldsymbol{L}_2\boldsymbol{D}_2\boldsymbol{U}_2=\boldsymbol{L}_1\boldsymbol{D}_1\boldsymbol{U}_1\\ \Longrightarrow\ &\boldsymbol{L}_1^{-1}(\boldsymbol{L}_2\boldsymbol{D}_2\boldsymbol{U}_2)\boldsymbol{U}_2^{-1}=\boldsymbol{L}_1^{-1}(\boldsymbol{L}_1\boldsymbol{D}_1\boldsymbol{U}_1)\boldsymbol{U}_2^{-1}\\ \Longrightarrow\ &\boldsymbol{L}_1^{-1}\boldsymbol{L}_2\boldsymbol{D}_2=\boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1}.\end{aligned}$$

By (i), $\boldsymbol{L}_1^{-1}$ is lower triangular with unit diagonal. By (ii), $\boldsymbol{L}_1^{-1}\boldsymbol{L}_2$ is lower triangular with unit diagonal. Therefore, by (iii), $\boldsymbol{L}_1^{-1}\boldsymbol{L}_2\boldsymbol{D}_2$ is lower triangular. Similarly, $\boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1}$ is upper triangular.

(b) Let $\boldsymbol{M}=\boldsymbol{L}_1^{-1}\boldsymbol{L}_2\boldsymbol{D}_2=\boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1}$. Then $\boldsymbol{M}$ is both lower and upper triangular, which implies that $\boldsymbol{M}$ is a diagonal matrix.

(1) Since $\boldsymbol{U}_1\boldsymbol{U}_2^{-1}$ has a unit diagonal, $\boldsymbol{M}=\boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1}$ has the same diagonal as $\boldsymbol{D}_1$. It implies that $\boldsymbol{M}=\boldsymbol{D}_1$. Similarly, we can have $\boldsymbol{M}=\boldsymbol{D}_2$. Therefore, $\boldsymbol{D}_1=\boldsymbol{D}_2$.

(2) For $\boldsymbol{M}=\boldsymbol{L}_1^{-1}\boldsymbol{L}_2\boldsymbol{D}_2=\boldsymbol{D}_2$, we have $\boldsymbol{L}_1^{-1}\boldsymbol{L}_2=\boldsymbol{I}$. Since the inverse matrix is unique, we have $\boldsymbol{L}_2=(\boldsymbol{L}_1^{-1})^{-1}=\boldsymbol{L}_1$.

(3) Similarly, for $\boldsymbol{M}=\boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1}=\boldsymbol{D}_1$, we have $\boldsymbol{U}_1\boldsymbol{U}_2^{-1}=\boldsymbol{I}$. It then implies that $\boldsymbol{U}_1=(\boldsymbol{U}_2^{-1})^{-1}=\boldsymbol{U}_2$.

8. (a) First do row exchange as

$$\boldsymbol{A} = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 2 & 2 \\ 2 & 6 & 7 \end{bmatrix} \overset{\boldsymbol{P}_{21}}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 2 & 6 & 7 \end{bmatrix} = \boldsymbol{PA}$$

and then perform elimination as

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 2 & 6 & 7 \end{bmatrix} \overset{\boldsymbol{E}_{31}}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \end{bmatrix} \overset{\boldsymbol{E}_{32}}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{U}.$$

Then we have

$$\boldsymbol{E}_{32}\boldsymbol{E}_{31}(\boldsymbol{PA}) = \boldsymbol{U}$$

where

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{E}_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{E}_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

We can have

$$\boldsymbol{L} = \boldsymbol{E}_{31}^{-1}\boldsymbol{E}_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}.$$

The factorization $\boldsymbol{PA} = \boldsymbol{LU}$ is hence given by

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 \\ 1 & 2 & 2 \\ 2 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) In order to factor $\boldsymbol{A}$ into $\boldsymbol{A} = \boldsymbol{L}_1\boldsymbol{P}_1\boldsymbol{U}_1$, we first perform elimination as

$$\boldsymbol{A} = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 2 & 2 \\ 2 & 6 & 7 \end{bmatrix} \overset{\boldsymbol{E}_{32}}{\Longrightarrow} \begin{bmatrix} 0 & 2 & 2 \\ 1 & 2 & 2 \\ 0 & 2 & 3 \end{bmatrix} \overset{\boldsymbol{E}_{31}}{\Longrightarrow} \begin{bmatrix} 0 & 2 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

and then do row exchange as

$$\begin{bmatrix} 0 & 2 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \overset{\boldsymbol{P}_{21}}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{U_1}.$$

Therefore,

$$\boldsymbol{U_1} = \boldsymbol{P}_{21}\boldsymbol{E}_{31}\boldsymbol{E}_{32}\boldsymbol{A}$$

where

$$\boldsymbol{P}_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{E}_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{E}_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}.$$

Multiplying $\boldsymbol{E}_{32}^{-1}\boldsymbol{E}_{31}^{-1}\boldsymbol{P}_{21}^{-1}$ from the left to both sides, we can have

$$\boldsymbol{A} = \boldsymbol{E}_{32}^{-1}\boldsymbol{E}_{31}^{-1}\boldsymbol{P}_{21}^{-1}\boldsymbol{U}_1 = \boldsymbol{L}_1\boldsymbol{P}_1\boldsymbol{U}_1$$

where

$$\boldsymbol{P}_1 = \boldsymbol{P}_{21}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\boldsymbol{L}_1 = \boldsymbol{E}_{32}^{-1}\boldsymbol{E}_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}.$$

The factorization $\boldsymbol{A} = \boldsymbol{L}_1\boldsymbol{P}_1\boldsymbol{U}_1$ is hence given by

$$\begin{bmatrix} 0 & 2 & 2 \\ 1 & 2 & 2 \\ 2 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$