EE 2030 Linear Algebra — Chi-chao Chao
Spring 2013

Homework Assignment No. 2
Due 10:10am, March 27, 2013

Reading: Strang, Chapter 3.

Problems for Solution:

1. Is each of the following subsets of $\mathcal{R}^3$ actually a subspace? ($\mathcal{R}$ is the set of real numbers.) If yes, prove it. Otherwise, find a counterexample.
   (a) All vectors (b_1, b_2, b_3) with $b_1 = 1$.
   (b) All vectors (b_1, b_2, b_3) satisfying $b_3 - b_2 + 3b_1 = 0$.
   (c) All linear combinations of $\boldsymbol{v} = (1, 1, 0)$ and $\boldsymbol{w} = (2, 0, 1)$.
2. Suppose S and T are two subspaces of a vector space V. The sum $S + T$ contains all sums $\boldsymbol{s} + \boldsymbol{t}$ of a vector $\boldsymbol{s}$ in S and a vector $\boldsymbol{t}$ in T, i.e.,
   $$S + T = \{\boldsymbol{s} + \boldsymbol{t} : \boldsymbol{s} \in S,\ \boldsymbol{t} \in T\}.$$
   The union $S \cup T$ contains all vectors from S or T or both, i.e.,
   $$S \cup T = \{\boldsymbol{v} : \boldsymbol{v} \in S \text{ or } \boldsymbol{v} \in T\}.$$
   Determine if each of the following statements is true. If it is, prove it. Otherwise, find a counterexample.
   (a) $S + T$ is a subspace of V.
   (b) $S \cup T$ is a subspace of V.
3. Do Problem 37 of Problem Set 3.2 in p. 143 of Strang.
4. Under what condition on b_1, b_2, b_3 is this system solvable? Find the complete solution when that condition holds:
   $$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$
5. Decide the dependence or independence of
   (a) $(1, 1, 2)$, $(1, 2, 1)$, $(3, 1, 1)$.
   (b) $\boldsymbol{v}_1 - \boldsymbol{v}_2$, $\boldsymbol{v}_2 - \boldsymbol{v}_3$, $\boldsymbol{v}_3 - \boldsymbol{v}_4$, $\boldsymbol{v}_4 - \boldsymbol{v}_1$ for any vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$, $\boldsymbol{v}_4$.

(c) $(1,1,0)$, $(1,0,0)$, $(0,1,1)$, (x,y,z) for any real numbers x, y, z.

6. Write down a matrix with the required property or explain why no such matrix exists.

   (a) Column space has basis $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; nullspace has basis $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

   (b) Column space contains $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$; row space contains $(1,1)$, $(1,2)$.

   (c) The only solution to $\boldsymbol{Ax} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$.

7. Find a basis for each of the four subspaces of the following matrices:

   (a)
$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & -2 & 0 & -1 \end{bmatrix}.$$

   (b)
$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 3 \end{bmatrix}.$$

   (*Hint:* Read about rank-one matrices in p. 145 and p. 189 of Strang.)

8. True or false. (If it is true, prove it. Otherwise, find a counterexample.)

   (a) If a square matrix $\boldsymbol{A}$ has independent columns, so does $\boldsymbol{A}^2$.

   (b) Suppose $\boldsymbol{A}$ is a 5 by 4 matrix with full column rank. The system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ is not solvable if the 5 by 5 matrix $[\,\boldsymbol{A}\ \boldsymbol{b}\,]$ is invertible.