# Solution to Homework Assignment No. 2

**1.** (a) No. Let $W = \{(b_1, b_2, b_3) : b_1 = 1\}$. Suppose $(1, b_2, b_3), (1, b_2', b_3') \in W$. Since

$$(1, b_2, b_3) + (1, b_2', b_3') = (2, b_2 + b_3', b_3 + b_3') \notin W$$

$W$ is not a subspace of $\mathcal{R}^3$

(b) Yes. Let $W = \{(b_1, b_2, b_3) : b_3 - b_2 + 3b_1 = 0\}$. Suppose $\boldsymbol{w}_1 = (b_1, b_2, b_3)$, $\boldsymbol{w}_2 = (b_1', b_2', b_3') \in W$. We check the following two conditions:

(i) Consider $\boldsymbol{w}_1 + \boldsymbol{w}_2 = (b_1 + b_1', b_2 + b_2', b_3 + b_3')$. Since

$$(b_3 + b_3') - (b_2 + b_2') + 3(b_1 + b_1') = (b_3 - b_2 + 3b_1) + (b_3' - b_2' + 3b_1') = 0$$

we have $\boldsymbol{w}_1 + \boldsymbol{w}_2 \in W$.

(ii) Consider $c\boldsymbol{w}_1 = (cb_1, cb_2, cb_3)$. Since $cb_3 - cb_2 + 3cb_1 = c(b_3 - b_2 + 3b_1) = 0$, we have $c\boldsymbol{w}_1 \in W$.

As a result, $W$ is a subspace of $\mathcal{R}^3$.

(c) Yes. Let $W = \{a_1(1,1,0) + a_2(2,0,1) : a_1, a_2 \in \mathcal{R}\}$. Suppose $\boldsymbol{w}_1 = a_1(1,1,0) + a_2(2,0,1)$, $\boldsymbol{w}_2 = a_1'(1,1,0) + a_2'(2,0,1) \in W$. We check the following two cases:

(i) Consider $\boldsymbol{w}_1 + \boldsymbol{w}_2$. We can have

$$\begin{aligned} \boldsymbol{w}_1 + \boldsymbol{w}_2 &= a_1(1,1,0) + a_2(2,0,1) + a_1'(1,1,0) + a_2'(2,0,1) \\ &= (a_1 + a_1')(1,1,0) + (a_2 + a_2')(2,0,1) \in W. \end{aligned}$$

(ii) Consider $c\boldsymbol{w}_1$ where $c \in \mathcal{R}$. Then we can obtain

$$\begin{aligned} c\boldsymbol{w}_1 &= c(a_1(1,1,0) + a_2(2,0,1)) \\ &= ca_1(1,1,0) + ca_2(2,0,1) \in W. \end{aligned}$$

Therefore, $W$ is a subspace of $\mathcal{R}^3$.

**2.** (a) True. Suppose $\boldsymbol{w}_1 = \boldsymbol{s}_1 + \boldsymbol{t}_1$, $\boldsymbol{w}_2 = \boldsymbol{s}_2 + \boldsymbol{t}_2 \in S + T$, where $\boldsymbol{s}_1, \boldsymbol{s}_2 \in S$ and $\boldsymbol{t}_1, \boldsymbol{t}_2 \in T$. Consider the following two conditions:

(i) $\boldsymbol{w}_1 + \boldsymbol{w}_2 = (\boldsymbol{s}_1 + \boldsymbol{t}_1) + (\boldsymbol{s}_2 + \boldsymbol{t}_2) = (\boldsymbol{s}_1 + \boldsymbol{s}_2) + (\boldsymbol{t}_1 + \boldsymbol{t}_2) \in S + T$ since $\boldsymbol{s}_1 + \boldsymbol{s}_2 \in S$ and $\boldsymbol{t}_1 + \boldsymbol{t}_2 \in T$.

(ii) $c\boldsymbol{w}_1 = c(\boldsymbol{s}_1 + \boldsymbol{t}_1) = c\boldsymbol{s}_1 + c\boldsymbol{t}_1 \in S + T$ since $c\boldsymbol{s}_1 \in S$ and $c\boldsymbol{t}_1 \in T$.

Therefore, $S + T$ is a subspace of $V$.

(b) False. Consider $S = \{(x, 0) : x \in \mathcal{R}\}$, $T = \{(0, y) : y \in \mathcal{R}\}$ are two subspaces of $\mathcal{R}^2$. Take $\boldsymbol{v} = (1,0) \in S$, $\boldsymbol{w} = (0,1) \in T$, and hence $\boldsymbol{v} \in S \cup T$, $\boldsymbol{w} \in S \cup T$. Since $\boldsymbol{v} + \boldsymbol{w} = (1,1) \notin S \cup T$, $S \cup T$ is not a subspace of $\mathcal{R}^2$.

**3.** From the figure, we can have

$$\begin{cases} y_1 + y_4 - y_3 &= 0 \\ y_2 + y_5 - y_1 &= 0 \\ y_3 + y_6 - y_2 &= 0 \\ y_4 + y_5 + y_6 &= 0 \end{cases} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \mathbf{0}.$$

Hence

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

By performing elimination on $\boldsymbol{A}$, we can obtain

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad \Longrightarrow \quad \boldsymbol{R} = \begin{bmatrix} 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The pivot variables are $y_1$, $y_2$, $y_4$ and the free variables are $y_3$, $y_5$, $y_6$. To have the special solutions, we let

$$\begin{aligned} y_3 = 1, y_5 = 0, y_6 = 0 &\Longrightarrow y_1 = 1, y_2 = 1, y_4 = 0 \\ y_3 = 0, y_5 = 1, y_6 = 0 &\Longrightarrow y_1 = 1, y_2 = 0, y_4 = -1 \\ y_3 = 0, y_5 = 0, y_6 = 1 &\Longrightarrow y_1 = 1, y_2 = 1, y_4 = -1. \end{aligned}$$

Hence, the special solutions are

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

**4.** By Gaussian elimination, we can have

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & b_1 \\ 2 & 6 & 9 & 5 & b_2 \\ -1 & -3 & 3 & 0 & b_3 \end{array}\right] \quad \Longrightarrow \quad \left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 3b_1 - b_2 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{2}{3}b_1 + \frac{1}{3}b_2 \\ 0 & 0 & 0 & 0 & 5b_1 - 2b_2 + b_3 \end{array}\right].$$

Hence the system is solvable if $5b_1 - 2b_2 + b_3 = 0$. Since the pivot variables are $x_1$, $x_3$ and the free variables are $x_2$, $x_4$, we can find a particular solution by letting

$$\begin{cases} x_2 = 0 \\ x_4 = 0 \end{cases} \quad \Longrightarrow \quad \begin{cases} x_1 = 3b_1 - b_2 \\ x_3 = -\frac{2}{3}b_1 + \frac{1}{3}b_2 \end{cases} \quad \Longrightarrow \quad \boldsymbol{x}_p = \begin{bmatrix} 3b_1 - b_2 \\ 0 \\ -\frac{2}{3}b_1 + \frac{1}{3}b_2 \\ 0 \end{bmatrix}.$$

Consider

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

To derive a general solution, we can have

$$\begin{array}{lll} x_2 = 1,\ x_4 = 0 & \Longrightarrow & x_1 = -3,\ \ x_3 = 0 \\ x_2 = 0,\ x_4 = 1 & \Longrightarrow & x_1 = -1,\ \ x_3 = -\frac{1}{3} \end{array} \Longrightarrow \boldsymbol{x}_n = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}.$$

As a result, the complete solution can be given as

$$\boldsymbol{x} = \boldsymbol{x}_n + \boldsymbol{x}_p = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} 3b_1 - b_2 \\ 0 \\ -\frac{2}{3}b_1 + \frac{1}{3}b_2 \\ 0 \end{bmatrix}$$

if $5b_1 - 2b_2 + b_3 = 0$.

**5.** (a) Suppose

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Since $\text{rank}(\boldsymbol{A}) = 3$, $\mathcal{N}(\boldsymbol{A}) = \{\boldsymbol{0}\}$. Hence $(1,1,2), (1,2,1), (3,1,1)$ are linearly independent.

(b) Since $(\boldsymbol{v}_1 - \boldsymbol{v}_2) + (\boldsymbol{v}_2 - \boldsymbol{v}_3) + (\boldsymbol{v}_3 - \boldsymbol{v}_4) + (\boldsymbol{v}_4 - \boldsymbol{v}_1) = \boldsymbol{0}$, they are linearly dependent.

(c) Since there are four vectors in $\mathcal{R}^3$, they must be linearly dependent.

**6.** (a) Since the column space and the nullspace both have three components, the desired matrix is 3 by 3, say $\boldsymbol{A}$. We can find that $\dim(\mathcal{N}(\boldsymbol{A})) = 1 \neq 2 = 3 - 1 = 3 - \text{rank}(\boldsymbol{A})$, which is not possible. Therefore, no such matrix exists.

(b) Consider the 3 by 2 matrix

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

We have that $\mathcal{C}(\boldsymbol{B})$ contains $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\mathcal{C}(\boldsymbol{B}^T)$ contains $(1,1) = (1,0) + (0,1)$ and $(1,2) = (1,0) + 2 \cdot (0,1)$.

(c) We can know that $\boldsymbol{A}$ must be 3 by 4. Since $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$ is the only solution to $\boldsymbol{Ax} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, the nullspace of $\boldsymbol{A}$ must contain the zero vector only. Hence, the rank of $\boldsymbol{A}$ should be 4. Yet as the number of rows of $\boldsymbol{A}$ is only 3, the rank of $\boldsymbol{A}$ cannot be 4. Therefore, $\boldsymbol{A}$ does not exist.

**7.** (a) Convert $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & -2 & 0 & -1 \end{bmatrix}$ into the RRE form:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & -2 & 0 & -1 \end{bmatrix} \implies \boldsymbol{R} = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, a basis for the row space of $\boldsymbol{A}$ can be given by

$$(1, 0, -2, 1),\ (0, 1, 1, 0).$$

The pivot columns are the 1st and 2nd columns of $\boldsymbol{R}$, and hence a basis for the column space of $\boldsymbol{A}$ can be given by

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

Since $x_1$ and $x_2$ are pivot variables and $x_3$ and $x_4$ are free variables, a basis for the nullspace of $\boldsymbol{A}$ can be given by the special solutions:

$$\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

We can have $\boldsymbol{R} = \boldsymbol{EA}$ where

$$\boldsymbol{E} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Since the last row of $\boldsymbol{R}$ is a zero row, a basis for the left nullspace of $\boldsymbol{A}$ can be given by the last row of $\boldsymbol{E}$:

$$(1, 0, 1).$$

(b) For the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 3 \end{bmatrix}$$

it is a rank-one matrix with the pivot row $(1, 0, 0, 3)$ and the pivot column $(1, 0, 2)^T$. Therefore, a basis for the row space of $\boldsymbol{A}$ is $(1, 0, 0, 3)$ and a basis for the column space of $\boldsymbol{A}$ is

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

For the nullspace of $\boldsymbol{A}$, we have

$$\begin{bmatrix} 1 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0.$$

Since $x_1$ is a pivot variable and $x_2$, $x_3$ and $x_4$ are free variables, a basis for the nullspace of $\boldsymbol{A}$ can be given by the special solutions:

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

For the left nullspace of $\boldsymbol{A}$, we have

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 0.$$

Therefore, a basis for the left nullspace of $\boldsymbol{A}$ can be given by

$$(0, 1, 0), \ (-2, 0, 1).$$

**8.** (a) True. Since a square matrix $\boldsymbol{A}$ has independent columns, it is full rank. That is to say that $\boldsymbol{A}$ has a full set of pivots. Therefore, $\boldsymbol{A}$ is invertible. Since $\boldsymbol{A}$ has $\boldsymbol{A}^{-1}$ as its inverse matrix, we have $(\boldsymbol{A}^{-1})^2$ as the inverse matrix for $\boldsymbol{A}^2$. This implies that $\boldsymbol{A}^2$ is also full rank. Therefore, $\boldsymbol{A}^2$ has independent columns.

(b) True. If the 5 by 5 matrix $\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \end{bmatrix}$ is invertible, there are 5 nonzero pivots and all the columns are independent. Hence $\boldsymbol{b}$ cannot be a linear combination of the columns of $\boldsymbol{A}$. Therefore, $\boldsymbol{Ax} = \boldsymbol{b}$ is not solvable.