EE 2030 Linear Algebra
Spring 2013

Chi-chao Chao

Homework Assignment No. 3
Due 10:10am, April 19, 2013

`Reading:` Strang, Chapter 4, Section 8.5.

`Problems for Solution:`

1. Do Problem 17 of Problem Set 4.1 in p. 204 of Strang.
2. (This problem counts double.) Consider
$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$
   (a) Find a basis for the orthogonal complement of the row space of $\boldsymbol{A}$.
   (b) Find the projection matrix $\boldsymbol{P}_C$ onto the column space of $\boldsymbol{A}$.
   (c) Find the projection matrix $\boldsymbol{P}_R$ onto the row space of $\boldsymbol{A}$.
   (d) Given $\boldsymbol{x} = (1, 2, 3)$, find $\boldsymbol{x}_r$ and $\boldsymbol{x}_n$ such that $\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n$, where $\boldsymbol{x}_r$ is in the row space of $\boldsymbol{A}$ and $\boldsymbol{x}_n$ is in the nullspace of $\boldsymbol{A}$.
   (e) Given
$$\boldsymbol{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$
   in the column space of $\boldsymbol{A}$, find a vector $\boldsymbol{x}_r$ in the row space of $\boldsymbol{A}$ such that
$$\boldsymbol{A}\boldsymbol{x}_r = \boldsymbol{b}.$$
3. (a) Find the best least squares fit by a plane $y = C + Dt + Ez$ to the four points
$$y = 3 \ \text{at} \ \ t = 1,\ z = 1 \qquad y = 6 \ \text{at} \ \ t = 0,\ z = 3$$
$$y = 5 \ \text{at} \ \ t = 2,\ z = 1 \qquad y = 0 \ \text{at} \ \ t = 0,\ z = 0.$$
   (b) Show that the best least squares fit to a set of measurements $y_1, y_2, \ldots, y_m$ by a horizontal line—in other words, by a constant function $y = C$—is their average
$$C = \frac{y_1 + y_2 + \cdots + y_m}{m}.$$
   (In statistical terms, the choice $\bar{y}$ that minimizes the error $E^2 = (y_1 - y)^2 + (y_2 - y)^2 + \cdots + (y_m - y)^2$ is the *mean* of the sample, and the resulting E^2 is the *variance* σ^2.)

4. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

   (a) Show that the partial derivatives of $\|\boldsymbol{Ax}\|^2$ with respect to $x_1, x_2, \ldots, x_n$ fill the vector $2\boldsymbol{A}^T\boldsymbol{Ax}$.

   (b) Show that the partial derivatives of $2\boldsymbol{b}^T\boldsymbol{Ax}$ fill the vector $2\boldsymbol{A}^T\boldsymbol{b}$.

   (c) Show that the partial derivatives of $\|\boldsymbol{Ax} - \boldsymbol{b}\|^2$ are zero when $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$.

5. Suppose $\boldsymbol{I}$ is the n by n identity matrix and $\boldsymbol{u}$ is an n by 1 unit vector, i.e., $\|\boldsymbol{u}\| = 1$. Consider the matrix $\boldsymbol{Q} = \boldsymbol{I} - 2\boldsymbol{uu}^T$.

   (a) Show that $\boldsymbol{Q}$ is an orthogonal matrix. (It is a reflection, also known as a *Householder transformation.*)

   (b) Show that $\boldsymbol{Qu} = -\boldsymbol{u}$.

   (c) Find $\boldsymbol{Qv}$ when $\boldsymbol{v}$ and $\boldsymbol{u}$ are orthogonal.

6. Consider

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

   (a) Apply the Gram-Schmidt process to find an orthonormal basis for the column space of $\boldsymbol{A}$.

   (b) Write $\boldsymbol{A}$ as $\boldsymbol{QR}$.

7. Consider the vector space $C[-1, 1]$, the space of all real-valued continuous functions on $[-1, 1]$, with inner product defined by

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

   (a) Find an orthonormal basis for the subspace spanned by 1, x, and x^2.

   (b) Find the best least squares approximation to x^3 on $[-1, 1]$ by a quadratic function $C + Dx + Ex^2$.