Solution to Homework Assignment No. 4

1. (a) By using row operations, we can obtain

$$|\boldsymbol{A}| = \begin{vmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 2 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 5 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 10 \end{vmatrix} = 1 \cdot -1 \cdot -2 \cdot 10 = 20$$

and

$$|\boldsymbol{B}| = \begin{vmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot 1 \cdot 1 \cdot 1 = 1.$$

(b) We have $|2\boldsymbol{A}| = 2^4 \cdot |\boldsymbol{A}| = 320$ and $|\boldsymbol{A}^T\boldsymbol{B}| = |\boldsymbol{A}^T||\boldsymbol{B}| = |\boldsymbol{A}||\boldsymbol{B}| = 20$.

2. (a) True. Since $\boldsymbol{Q}$ is an orthogonal matrix, we have $\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}$. We can then obtain $1 = |\boldsymbol{I}| = |\boldsymbol{Q}^T\boldsymbol{Q}| = |\boldsymbol{Q}^T||\boldsymbol{Q}| = |\boldsymbol{Q}||\boldsymbol{Q}| = |\boldsymbol{Q}|^2$. Therefore, $\det \boldsymbol{Q}$ is equal to 1 or -1.

(b) True. Since $\boldsymbol{A}$ is not invertible, we have $|\boldsymbol{A}| = 0$. Then we can obtain $|\boldsymbol{AB}| = |\boldsymbol{A}|\,|\boldsymbol{B}| = 0 \cdot |\boldsymbol{B}| = 0$. Hence $\boldsymbol{AB}$ is not invertible.

(c) False. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. We have $|\boldsymbol{A} - \boldsymbol{B}| = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1$. However, $|\boldsymbol{A}| - |\boldsymbol{B}| = 0 - 0 = 0$. Hence $|\boldsymbol{A} - \boldsymbol{B}| \neq |\boldsymbol{A}| - |\boldsymbol{B}|$, which gives a counterexample.

(d) False. Let $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. We have $\boldsymbol{A}^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -\boldsymbol{A}$. It is skew-symmetric. However, $\det \boldsymbol{A} = 1$, which gives a counterexample.

3. Let $F_n = |\boldsymbol{A_n}|$, where $\boldsymbol{A_n}$ is an n by n matrix. For $n \geq 3$, we have

$$F_n = \begin{vmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & & & & \\ 0 & & \boldsymbol{A_{n-1}} & & \\ \vdots & & & & \\ 0 & & & & \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 0 \cdots & 0 \\ 1 & 1 & -1 & 0 \cdots & 0 \\ 0 & 1 & & & \\ 0 & 0 & & \boldsymbol{A_{n-2}} & \\ \vdots & \vdots & & & \\ 0 & 0 & & & \end{vmatrix}.$$

Applying the cofactor formula to the first row, we can have

$$\begin{aligned} F_n &= 1 \cdot (-1)^{1+1} |\boldsymbol{A_{n-1}}| + (-1) \cdot (-1)^{1+2} \begin{vmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & & & & \\ 0 & & & \boldsymbol{A_{n-2}} & \\ \vdots & & & & \\ 0 & & & & \end{vmatrix} \\ &= F_{n-1} + 1 \cdot (-1)^{1+1} |\boldsymbol{A_{n-2}}| \ \text{(apply the cofactor formula to the first column)} \\ &= F_{n-1} + F_{n-2}. \end{aligned}$$

4. (a) Let $S_n = |\boldsymbol{A_n}|$, where $\boldsymbol{A_n}$ is an n by n matrix. For $n \geq 3$, we have

$$S_n = \begin{vmatrix} 3 & 1 & 0 & \cdots & 0 \\ 1 & & & & \\ 0 & & \boldsymbol{A_{n-1}} & & \\ \vdots & & & & \\ 0 & & & & \end{vmatrix} = \begin{vmatrix} 3 & 1 & 0 & 0 \cdots & 0 \\ 1 & 3 & 1 & 0 \cdots & 0 \\ 0 & 1 & & & \\ 0 & 0 & & \boldsymbol{A_{n-2}} & \\ \vdots & \vdots & & & \\ 0 & 0 & & & \end{vmatrix}.$$

Applying the cofactor formula to the first row, we can have

$$\begin{aligned} S_n &= 3 \cdot (-1)^{1+1} |\boldsymbol{A_{n-1}}| + 1 \cdot (-1)^{1+2} \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & & & & \\ 0 & & & \boldsymbol{A_{n-2}} & \\ \vdots & & & & \\ 0 & & & & \end{vmatrix} \\ &= 3S_{n-1} - 1 \cdot (-1)^{1+1} |\boldsymbol{A_{n-2}}| \ \text{(apply the cofactor formula to the first column)} \\ &= 3S_{n-1} - S_{n-2}. \end{aligned}$$

Therefore, we can obtain $a = 3$ and $b = -1$.

(b) We have

$$\begin{aligned} S_1 &= 3 \\ S_2 &= 8 \\ S_3 &= 3S_2 - S_1 = 21 \\ S_4 &= 3S_3 - S_2 = 55 \\ S_5 &= 3S_4 - S_3 = 144. \end{aligned}$$

5. (a) Consider the last three rows

$$\begin{bmatrix} 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}.$$

The rank of this submatrix is at most 2. Therefore, the rows are dependent.

(b) The big formula states that the determinant of $\boldsymbol{A}$ is the sum of 5! simple determinants, times 1 or -1, and every simple determinant chooses one entry from each row and column. From the last three rows, we can see that if some simple determinant of $\boldsymbol{A}$ avoids all the zero entries in $\boldsymbol{A}$, then it cannot choose one entry from each column. Thus every simple determinant of $\boldsymbol{A}$ must choose at least one zero entry, and hence all the terms are zero in the big formula for $\det \boldsymbol{A}$.

6. (a) For the first system, we have

$$\begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Using Cramer's rule, we can obtain

$$x_1 = \frac{\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix}} = -2 \text{ and } x_2 = \frac{\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix}} = 1.$$

(b) For the second system, we have

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Using Cramer's rule, we can obtain

$$x_1 = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}} = \frac{3}{4},\ x_2 = \frac{\begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}} = -\frac{1}{2}, \text{ and } x_3 = \frac{\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}} = \frac{1}{4}.$$

7. (a) We have

$$\boldsymbol{C}_{11} = \begin{vmatrix} 2 & 2 \\ 2 & 4 \end{vmatrix} = 4,\ \boldsymbol{C}_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = 3,\ \boldsymbol{C}_{33} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$$

$$\boldsymbol{C}_{21} = \boldsymbol{C}_{12} = -\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = -2,\ \boldsymbol{C}_{31} = \boldsymbol{C}_{13} = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0,\ \boldsymbol{C}_{32} = \boldsymbol{C}_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1.$$

Therefore, the cofactor matrix is given by

$$\boldsymbol{C} = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

(b) Since $(\boldsymbol{C}^T / \det \boldsymbol{A}) = \boldsymbol{A}^{-1}$, we know that $\boldsymbol{A}\boldsymbol{C}^T = (\det \boldsymbol{A})\boldsymbol{A}\boldsymbol{A}^{-1} = (\det \boldsymbol{A})\boldsymbol{I}$. We have

$$\boldsymbol{A}\boldsymbol{C}^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = (\det \boldsymbol{A})\boldsymbol{I}.$$

Therefore, it can be obtained that $\det \boldsymbol{A} = 2$.

8. (a) Since the Hadamard matrix $\boldsymbol{H}_4$ has orthogonal rows, the box is a hypercube and the absolute value of the volume is the multiplication of lengths of the row vectors. We know that every row vector has equal length $\sqrt{1^2+1^2+1^2+1^2} = 2$. Therefore,

$$|\det \boldsymbol{H}_4| = 2 \cdot 2 \cdot 2 \cdot 2 = 16.$$

(b) For $\boldsymbol{H}_4$, from (a) we can have $\boldsymbol{H}_4\boldsymbol{H}_4^T = 4\boldsymbol{I}_4$, where $\boldsymbol{I}_n$ is the n by n identity matrix. We now have

$$\begin{aligned}
\boldsymbol{H}_8\boldsymbol{H}_8^T &= \begin{bmatrix} \boldsymbol{H}_4 & \boldsymbol{H}_4 \\ \boldsymbol{H}_4 & -\boldsymbol{H}_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{H}_4^T & \boldsymbol{H}_4^T \\ \boldsymbol{H}_4^T & -\boldsymbol{H}_4^T \end{bmatrix} \\
&= \begin{bmatrix} 2\boldsymbol{H}\boldsymbol{H}_4^T & \boldsymbol{H}_4\boldsymbol{H}_4^T - \boldsymbol{H}_4\boldsymbol{H}_4^T \\ \boldsymbol{H}_4\boldsymbol{H}_4^T - \boldsymbol{H}_4\boldsymbol{H}_4^T & 2\boldsymbol{H}\boldsymbol{H}_4^T \end{bmatrix} \\
&= \begin{bmatrix} 8\boldsymbol{I}_4 & \boldsymbol{O} \\ \boldsymbol{O} & 8\boldsymbol{I}_4 \end{bmatrix} = 8\boldsymbol{I}_8.
\end{aligned}$$

Therefore, the rows of $\boldsymbol{H}_8$ are mutually orthogonal. It is still a hypercube and the absolute value of the volume is the multiplication of lengths of the row vectors. Every row vector has equal length $\sqrt{8}$. Therefore,

$$|\det \boldsymbol{H}_8| = \left(\sqrt{8}\right)^8 = 4096.$$