EE 2030 Linear Algebra Chi-chao Chao
Spring 2013

Homework Assignment No. 5
Due 10:10am, May 31, 2013

Reading: Strang, Chapter 6, Handout "Spectral Theorem."

Problems for Solution:

1. Prove each of the following statements:

   (a) If λ is an eigenvalue of an invertible matrix $\boldsymbol{A}$, then λ^{-1} is an eigenvalue of $\boldsymbol{A}^{-1}$.

   (b) The eigenvalues of $\boldsymbol{A}$ are the same as the eigenvalues of $\boldsymbol{A}^T$.

   (c) If λ is an eigenvalue of an *idempotent* matrix, then λ must be either 0 or 1. (An n by n matrix $\boldsymbol{A}$ is said to be idempotent if $\boldsymbol{A}^2 = \boldsymbol{A}$.)

2. Determine if each of the following matrices is diagonalizable. If it is, find an invertible matrix $\boldsymbol{S}$ and a diagonal matrix $\boldsymbol{\Lambda}$ such that $\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = \boldsymbol{\Lambda}$. If it is not, find its Jordan form.

   (a) $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$.

   (b) $\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

3. (a) Do Problem 19 of Problem Set 6.2 in p. 309 of Strang.

   (b) Do Problem 36 of Problem Set 6.2 in p. 311 of Strang.

4. (a) Solve the difference equation $G_{k+2} - (1/2)G_{k+1} - (1/2)G_k = 0$ with initial conditions $G_0 = 0$ and $G_1 = 1/2$.

   (b) Solve the differential equation $y'' - 5y' + 4y = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 3$.

5. (This problem counts double.) Determine if each of the following statements is true. If yes, prove it. Otherwise, show why it is not or find a counterexample.

   (a) The eigenvalues of a negative definite matrix are all negative.

   (b) If $\boldsymbol{A}$ is positive definite, then $\boldsymbol{A}^{-1}$ is positive definite.

   (c) A positive definite matrix is always invertible.

   (d) $\begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix}$ is positive definite.

(e) If $\boldsymbol{A}$ is a symmetric nonsingular matrix, then $\boldsymbol{A}^2$ is positive definite.

(f) If $\boldsymbol{B}^2$ is similar to $\boldsymbol{A}^2$, then $\boldsymbol{B}$ is similar to $\boldsymbol{A}$.

6. Which of these six matrices are similar?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

7. Consider

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

(a) Compute $\boldsymbol{A}^T\boldsymbol{A}$ and $\boldsymbol{A}\boldsymbol{A}^T$. Then find their eigenvalues and unit eigenvectors.

(b) Construct the singular value decomposition and verify that $\boldsymbol{A}$ equals $\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$.

(c) Find orthonormal bases for the four fundamental subspaces of $\boldsymbol{A}$.