Solution to Homework Assignment No. 5

1. (a) Let $\boldsymbol{x}$ be the associated eigenvector of λ. We have

$$\begin{aligned} & \boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x} \\ \Longrightarrow\quad & \boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^{-1}\lambda\boldsymbol{x} \\ \Longrightarrow\quad & \boldsymbol{I}\boldsymbol{x} = \lambda\boldsymbol{A}^{-1}\boldsymbol{x} \\ \Longrightarrow\quad & \boldsymbol{A}^{-1}\boldsymbol{x} = \lambda^{-1}\boldsymbol{x}. \end{aligned}$$

Hence, λ^{-1} is an eigenvalue of $\boldsymbol{A}^{-1}$.

(b) Let λ be a eigenvalue of $\boldsymbol{A}$. Since

$$\begin{aligned} & \det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0 \\ \Longleftrightarrow\quad & \det((\boldsymbol{A} - \lambda\boldsymbol{I})^T) = 0 \\ \Longleftrightarrow\quad & \det(\boldsymbol{A}^T - \lambda\boldsymbol{I}^T) = 0 \\ \Longleftrightarrow\quad & \det(\boldsymbol{A}^T - \lambda\boldsymbol{I}) = 0 \end{aligned}$$

we can obtain that λ is also an eigenvalue of $\boldsymbol{A}^T$, and vice versa.

(c) Let $\boldsymbol{x}$ be the associated eigenvector of λ. Since $\boldsymbol{A}$ is idempotent, we have

$$\begin{aligned} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^2\boldsymbol{x} = \boldsymbol{A}(\boldsymbol{A}\boldsymbol{x}) = \boldsymbol{A}(\lambda\boldsymbol{x}) = \lambda(\boldsymbol{A}\boldsymbol{x}) = \lambda^2\boldsymbol{x} = \lambda\boldsymbol{x} \\ & \Longrightarrow (\lambda^2 - \lambda)\boldsymbol{x} = \boldsymbol{0}. \end{aligned}$$

Since $\boldsymbol{x}$ is not a zero vector, we must have $\lambda^2 - \lambda = 0$, i.e., $\lambda = 0$ or 1.

2. (a) Consider

$$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & -2 \\ 2 & -2 & -\lambda \end{vmatrix} = \lambda(3-\lambda)(3+\lambda) = 0.$$

We can then find that the eigenvalues of $\boldsymbol{A}$ are $\lambda = 0, 3, -3$. Since the eigenvalues are all distinct, $\boldsymbol{A}$ is diagonalizable. We can obtain that the eigenvectors for $\lambda = 0, 3, -3$ are $\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$, respectively. Therefore, let

$$\boldsymbol{S} = \begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Lambda} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

and then $\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = \boldsymbol{\Lambda}$.

(b) Consider

$$\det(\boldsymbol{A}-\lambda\boldsymbol{I}) = \begin{vmatrix} -\lambda & 1 & 2 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda^3 = 0.$$

We can obtain $\lambda = 0$, and its AM is 3. Since $\boldsymbol{A}-\lambda\boldsymbol{I} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the GM of λ is 2, which is small than the AM of λ. Therefore, this matrix is not diagonalizable, and its Jordan form is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

3. (a) First, we find the eigenvalues of $\boldsymbol{B}$ by

$$\det(\boldsymbol{B}-\lambda\boldsymbol{I}) = \det\left(\begin{bmatrix} 5-\lambda & 1 \\ 0 & 4-\lambda \end{bmatrix}\right) = (5-\lambda)(4-\lambda) = 0.$$

Hence $\lambda = 5$ and 4. For $\lambda = 5$, we have

$$(\boldsymbol{B}-\lambda\boldsymbol{I})\,\boldsymbol{x}_1 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\boldsymbol{x}_1 = \boldsymbol{0}$$

and the eigenvector $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. For $\lambda = 4$, we have

$$(\boldsymbol{B}-\lambda\boldsymbol{I})\,\boldsymbol{x}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{x}_2 = \boldsymbol{0}$$

and the eigenvector $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Hence, we have

$$\boldsymbol{B} = \boldsymbol{S}\boldsymbol{A}\boldsymbol{S}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

and

$$\begin{aligned} \boldsymbol{B}^k = \left(\boldsymbol{S}\boldsymbol{A}\boldsymbol{S}^{-1}\right)^k = \boldsymbol{S}\boldsymbol{A}^k\boldsymbol{S}^{-1} &= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 5^k & 0 \\ 0 & 4^k \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 5^k & 5^k \\ 0 & -4^k \end{bmatrix} \\ &= \begin{bmatrix} 5^k & 5^k-4^k \\ 0 & 4^k \end{bmatrix}. \end{aligned}$$

(b) Assume λ_1 and λ_2 are eigenvalues of $\begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\begin{bmatrix} i \\ 1 \end{bmatrix}$, respectively. We can then have

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = \lambda_1\begin{bmatrix} 1 \\ i \end{bmatrix} \implies \begin{bmatrix} \cos\theta - i\sin\theta \\ \sin\theta + i\cos\theta \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ i\lambda_1 \end{bmatrix}$$

$$\implies \lambda_1 = \cos\theta - i\sin\theta$$

and

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} = \lambda_2\begin{bmatrix} i \\ 1 \end{bmatrix} \implies \begin{bmatrix} i\cos\theta - \sin\theta \\ i\sin\theta + \cos\theta \end{bmatrix} = \begin{bmatrix} i\lambda_2 \\ \lambda_2 \end{bmatrix}$$

$$\implies \lambda_2 = \cos\theta + i\sin\theta.$$

Hence

$$\boldsymbol{A} = \boldsymbol{S\Lambda S}^{-1} = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} \cos\theta - i\sin\theta & 0 \\ 0 & \cos\theta + i\sin\theta \end{bmatrix}\begin{bmatrix} 1/2 & -i/2 \\ -i/2 & 1/2 \end{bmatrix}$$

and

$$\begin{aligned}
\boldsymbol{A}^n &= \boldsymbol{S\Lambda}^n\boldsymbol{S}^{-1} \\
&= \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} \cos\theta - i\sin\theta & 0 \\ 0 & \cos\theta + i\sin\theta \end{bmatrix}^n\begin{bmatrix} 1/2 & -i/2 \\ -i/2 & 1/2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}^n\begin{bmatrix} 1/2 & -i/2 \\ -i/2 & 1/2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} e^{-in\theta} & 0 \\ 0 & e^{in\theta} \end{bmatrix}\begin{bmatrix} 1/2 & -i/2 \\ -i/2 & 1/2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}e^{-in\theta} + \frac{1}{2}e^{in\theta} & \frac{-i}{2}e^{-in\theta} + \frac{i}{2}e^{in\theta} \\ \frac{i}{2}e^{-in\theta} + \frac{-i}{2}e^{in\theta} & \frac{1}{2}e^{-in\theta} + \frac{1}{2}e^{in\theta} \end{bmatrix} \\
&= \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.
\end{aligned}$$

4. (a) Assume $\boldsymbol{u}_k = \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$. Then $\boldsymbol{u}_0 = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$ and

$$\boldsymbol{u}_{k+1} = \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} (1/2)G_{k+1} + (1/2)G_k \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}\boldsymbol{u}_k.$$

Let $\boldsymbol{A} = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$. We then diagonalize $\boldsymbol{A} = \boldsymbol{S\Lambda S}^{-1}$ where

$$\boldsymbol{S} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}, \quad \boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}, \quad \boldsymbol{S}^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

Hence, we can obtain

$$\boldsymbol{u}_k = \boldsymbol{A}^k\boldsymbol{u}_0 = \boldsymbol{S\Lambda}^k\boldsymbol{S}^{-1}\boldsymbol{u}_0 = \begin{bmatrix} (1/3) + (1/6)(-1/2)^k \\ (1/3) - (1/3)(-1/2)^k \end{bmatrix}$$

and

$$G_k = (1/3) - (1/3)(-1/2)^k.$$

(b) Assume $\boldsymbol{u} = \begin{bmatrix} y' \\ y \end{bmatrix}$, $\boldsymbol{u}' = \begin{bmatrix} y'' \\ y' \end{bmatrix}$, and $\boldsymbol{u}_0 = \begin{bmatrix} y'(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Then we can have

$$\boldsymbol{u}' = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \begin{bmatrix} 5y' - 4y \\ y' \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 1 & 0 \end{bmatrix}\boldsymbol{u}.$$

Let $\boldsymbol{A} = \begin{bmatrix} 5 & -4 \\ 1 & 0 \end{bmatrix}$. We then diagonalize $\boldsymbol{A} = \boldsymbol{S\Lambda S}^{-1}$ where

$$\boldsymbol{S} = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}, \quad \boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad \boldsymbol{S}^{-1} = \frac{-1}{3} \begin{bmatrix} 1 & -4 \\ -1 & 1 \end{bmatrix}.$$

Hence, we can obtain

$$\boldsymbol{u} = e^{\boldsymbol{A}t}\boldsymbol{u}_0 = \boldsymbol{S}e^{\boldsymbol{\Lambda}t}\boldsymbol{S}^{-1}\boldsymbol{u}_0 = \begin{bmatrix} -e^t + 4e^{4t} \\ -e^t + e^{4t} \end{bmatrix}$$

and

$$y = -e^t + e^{4t}.$$

5. (a) True. Assume $\boldsymbol{A}$ is a negative definite matrix. Let λ be an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{x}$ is a corresponding unit eigenvector. Then we can have

$$\boldsymbol{Ax} = \lambda\boldsymbol{x} \quad \Longrightarrow \quad \boldsymbol{x}^T\boldsymbol{Ax} = \boldsymbol{x}^T\lambda\boldsymbol{x} = \lambda\|\boldsymbol{x}\|^2 = \lambda < 0.$$

(b) True. By part (a) of Problem 1, λ^{-1} is an eigenvalue of $\boldsymbol{A}^{-1}$ if λ is an eigenvalue of $\boldsymbol{A}$. Since all λ's are positive for a positive definite matrix $\boldsymbol{A}$, all λ^{-1}'s are positive. Hence, $\boldsymbol{A}^{-1}$ is positive definite.

(c) True. Since the determinant of a positive definite matrix is positive, this matrix is invertible.

(d) False. Since the determinant of this matrix is negative, it is not positive definite.

(e) True. Since $\boldsymbol{A}$ is nonsingular, all the eigenvalues of $\boldsymbol{A}$ are nonzero. We learned from class that λ^2 is an eigenvalue of $\boldsymbol{A}^2$ if λ is an eigenvalue of $\boldsymbol{A}$. Therefore, all the eigenvalues of $\boldsymbol{A}^2$ are positive, which implies that $\boldsymbol{A}^2$ is positive definite.

(f) False. Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Since $\boldsymbol{A}^2 = \boldsymbol{B}^2 = \boldsymbol{I}_2$, $\boldsymbol{B}^2$ is similar to $\boldsymbol{A}^2$. However, since the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$ are not the same, $\boldsymbol{B}$ is not similar to $\boldsymbol{A}$.

6. We can find the eigenvalues of each matrix as follows.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} : \ \lambda = 1, 1$$
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : \ \lambda = 1, -1$$
$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} : \ \lambda = 0, 1$$
$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} : \ \lambda = 0, 1$$
$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} : \ \lambda = 0, 1$$
$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} : \ \lambda = 0, 1.$$

Since all 2×2 matrices with eigenvalues 0 and 1 are similar to each other (as they are all similar to a diagonal matrix with 0, 1 on the diagonal), the following matrices are similar:

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

The matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

are similar to themselves.

7. (a) After some calculations, we can obtain the eigenvalues and unit eigenvectors of $\boldsymbol{A}^T\boldsymbol{A}$ and $\boldsymbol{A}\boldsymbol{A}^T$ as follows:

$$\boldsymbol{A}^T\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \Longrightarrow \quad \begin{cases} \lambda_1 = 3 & \longleftrightarrow & \boldsymbol{v}_1 = \frac{1}{\sqrt{2}}(1, 1)^T \\ \lambda_2 = 1 & \longleftrightarrow & \boldsymbol{v}_2 = \frac{1}{\sqrt{2}}(1, -1)^T. \end{cases}$$

$$\boldsymbol{A}\boldsymbol{A}^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \Longrightarrow \quad \begin{cases} \lambda_1 = 3 & \longleftrightarrow & \boldsymbol{u}_1 = \frac{1}{\sqrt{6}}(1, 2, 1)^T \\ \lambda_2 = 1 & \longleftrightarrow & \boldsymbol{u}_2 = \frac{1}{\sqrt{2}}(1, 0, -1)^T \\ \lambda_3 = 0 & \longleftrightarrow & \boldsymbol{u}_3 = \frac{1}{\sqrt{3}}(1, -1, 1)^T. \end{cases}$$

(b) According to (a), the singular value decomposition of $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$$

where

$$\boldsymbol{U} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & \sqrt{3} & \sqrt{2} \\ 2 & 0 & -\sqrt{2} \\ 1 & -\sqrt{3} & \sqrt{2} \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{V} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The decomposition can be verified by

$$\begin{aligned}
\boldsymbol{U\Sigma V}^T &= \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & \sqrt{3} & \sqrt{2} \\ 2 & 0 & -\sqrt{2} \\ 1 & -\sqrt{3} & \sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^T \\
&= \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & \sqrt{3} & \sqrt{2} \\ 2 & 0 & -\sqrt{2} \\ 1 & -\sqrt{3} & \sqrt{2} \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 1 & -1 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \boldsymbol{A}.
\end{aligned}$$

(c) According to what was taught in class, we know that we can use the unit eigenvectors obtained in (a) to form orthonormal bases for the four fundamental subspaces of $\boldsymbol{A}$. Therefore, $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, $\{\boldsymbol{u}_3\}$, $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, and ϕ are orthonormal bases for $\mathcal{C}(\boldsymbol{A})$, $\mathcal{N}(\boldsymbol{A}^T)$, $\mathcal{C}(\boldsymbol{A}^T)$, and $\mathcal{N}(\boldsymbol{A})$, respectively. Note that the basis for the nullspace is the empty set.