Solution to Homework Assignment No. 6

1. (a) No. Let $\boldsymbol{v} = (1, 0)$ and $\boldsymbol{w} = (0, 1)$. Since

$$\begin{array}{rcl} T(\boldsymbol{v}) = (1,0),\ \ T(\boldsymbol{w}) = (0,1) & \Longrightarrow & T(\boldsymbol{v}) + T(\boldsymbol{w}) = (1,1) \\ \boldsymbol{v} + \boldsymbol{w} = (1,1) & \Longrightarrow & T(\boldsymbol{v} + \boldsymbol{w}) = (1/\sqrt{2}, 1/\sqrt{2}) \end{array}$$

we have $T(\boldsymbol{v}) + T(\boldsymbol{w}) \neq T(\boldsymbol{v} + \boldsymbol{w})$ and T is not a linear transformation.

(b) Yes. Let $\boldsymbol{v} = (v_1, v_2, v_3)$ and $\boldsymbol{w} = (w_1, w_2, w_3)$. Then

$$\begin{aligned} & T(c\boldsymbol{v} + d\boldsymbol{w}) \\ =\ & T(cv_1 + dw_1, cv_2 + dw_2, cv_3 + dw_3) \\ =\ & (cv_1 + dw_1, 2cv_2 + 2dw_2, 3cv_3 + 3dw_3) \\ =\ & c(v_1, 2v_2, 3v_3) + d(w_1, 2w_2, 3w_3) \\ =\ & cT(\boldsymbol{v}) + dT(\boldsymbol{w}). \end{aligned}$$

Hence, T is a linear transformation.

(c) No. Let $\boldsymbol{v} = (1, 0)$ and $\boldsymbol{w} = (0, 1)$. Since

$$\begin{array}{rcl} T(\boldsymbol{v}) = 1,\ \ T(\boldsymbol{w}) = 1 & \Longrightarrow & T(\boldsymbol{v}) + T(\boldsymbol{w}) = 2 \\ \boldsymbol{v} + \boldsymbol{w} = (1,1) & \Longrightarrow & T(\boldsymbol{v} + \boldsymbol{w}) = 1 \end{array}$$

we have $T(\boldsymbol{v}) + T(\boldsymbol{w}) \neq T(\boldsymbol{v} + \boldsymbol{w})$ and T is not a linear transformation.

2. (a) Let $\boldsymbol{X}, \boldsymbol{Y} \in M$. Then we can have

$$T(a\boldsymbol{X} + b\boldsymbol{Y}) = \boldsymbol{A}(a\boldsymbol{X} + b\boldsymbol{Y}) = a\boldsymbol{A}\boldsymbol{X} + b\boldsymbol{A}\boldsymbol{Y} = aT(\boldsymbol{X}) + bT(\boldsymbol{Y})$$

and hence T is linear.

(b) Since $\beta = \{\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3, \boldsymbol{V}_4\}$ and

$$\begin{aligned} T(\boldsymbol{V}_1) &= \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = a\boldsymbol{V}_1 + c\boldsymbol{V}_3 \\ T(\boldsymbol{V}_2) &= \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = a\boldsymbol{V}_2 + c\boldsymbol{V}_4 \\ T(\boldsymbol{V}_3) &= \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix} = b\boldsymbol{V}_1 + d\boldsymbol{V}_3 \\ T(\boldsymbol{V}_4) &= \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} = b\boldsymbol{V}_2 + d\boldsymbol{V}_4 \end{aligned}$$

we can have

$$[T]_\beta = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}.$$

3. (a) Since $\beta = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, $\gamma = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$, and

$$\begin{aligned} T(\boldsymbol{v}_1) &= \boldsymbol{w}_2 \\ T(\boldsymbol{v}_2) &= \boldsymbol{w}_1 + \boldsymbol{w}_3 \\ T(\boldsymbol{v}_3) &= \boldsymbol{w}_1 + \boldsymbol{w}_3 \end{aligned}$$

we can obtain

$$[T]_\beta^\gamma = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

(b) From (a), we can find that $\{(0, 1, -1)^T\}$ forms a basis for $\mathcal{N}([T]_\beta^\gamma)$. Therefore, the kernel of T is given by the span of $\boldsymbol{v}_2 - \boldsymbol{v}_3$.

(c) The dimension of the range of T is equal to that of the column space of $[T]_\beta^\gamma$, which is 2.

4. Assume $T : V \to W$. Since $T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v}$ and

- $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is a basis for $\mathcal{C}(\boldsymbol{A}^T)$
- $\{\boldsymbol{v}_3, \boldsymbol{v}_4\}$ is a basis for $\mathcal{N}(\boldsymbol{A})$
- $\{\boldsymbol{w}_1, \boldsymbol{w}_2\}$ is a basis for $\mathcal{C}(\boldsymbol{A})$
- $\{\boldsymbol{w}_3\}$ is a basis for $\mathcal{N}(\boldsymbol{A}^T)$

we can have $\beta = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ and $\gamma = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$ form a basis for V and W, respectively. As

$$\begin{aligned} T(\boldsymbol{v}_1) &= \boldsymbol{A}\boldsymbol{v}_1 = \boldsymbol{w}_1 \\ T(\boldsymbol{v}_2) &= \boldsymbol{A}\boldsymbol{v}_2 = \boldsymbol{w}_2 \\ T(\boldsymbol{v}_3) &= \boldsymbol{0} \\ T(\boldsymbol{v}_4) &= \boldsymbol{0} \end{aligned}$$

we can then obtain

$$[T]_\beta^\gamma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

5. Based on the standard basis, the matrix which represents this T is

$$\begin{bmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{bmatrix}.$$

The eigenvectors for this matrix are $\{(-1, -2, 1), (-2, 1, 0), (1, 0, 1)\}$. Therefore, we can find the basis $\{(-1, -2, 1), (-2, 1, 0), (1, 0, 1)\}$ in which the matrix representation for T is a diagonal matrix.

6. (a) For all $\boldsymbol{x} \in \mathbb{R}^n$, we can have

$$\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n$$

where $\boldsymbol{x}_r \in \mathcal{C}(\boldsymbol{A}^T)$ and $\boldsymbol{x}_n \in \mathcal{N}(\boldsymbol{A})$. We then have, for all $\boldsymbol{x} \in \mathbb{R}^n$,

$$\begin{aligned}
\boldsymbol{A}\boldsymbol{A}^+\boldsymbol{A}\boldsymbol{x} &= \boldsymbol{A}\boldsymbol{A}^+\boldsymbol{A}(\boldsymbol{x}_r + \boldsymbol{x}_n) \\
&= \boldsymbol{A}\boldsymbol{A}^+\boldsymbol{A}\boldsymbol{x}_r + \boldsymbol{A}\boldsymbol{A}^+\boldsymbol{A}\boldsymbol{x}_n \\
&= \boldsymbol{A}(\boldsymbol{x}_r) + \boldsymbol{A}\boldsymbol{A}^+\boldsymbol{0} \\
&= \boldsymbol{A}\boldsymbol{x}_r \\
&= \boldsymbol{A}(\boldsymbol{x}_r + \boldsymbol{x}_n) \\
&= \boldsymbol{A}\boldsymbol{x}
\end{aligned}$$

since $\boldsymbol{A}^+\boldsymbol{A}$ is the projection matrix onto $\mathcal{C}(\boldsymbol{A}^T)$ and $\boldsymbol{A}\boldsymbol{x}_n = \boldsymbol{0}$. Therefore, $\boldsymbol{A}\boldsymbol{A}^+\boldsymbol{A} = \boldsymbol{A}$

(b) For a projection matrix $\boldsymbol{P}$, we have $\boldsymbol{P}^2 = \boldsymbol{P}$. Since $\boldsymbol{A}^+\boldsymbol{A}$ is the projection matrix onto $\mathcal{C}(\boldsymbol{A}^T)$,

$$(\boldsymbol{A}^+\boldsymbol{A})^2 = \boldsymbol{A}^+\boldsymbol{A}.$$

7. (a) After some calculations, we can obtain the eigenvalues and unit eigenvectors of $\boldsymbol{A}^T\boldsymbol{A}$ and $\boldsymbol{A}\boldsymbol{A}^T$ as follows:

$$\boldsymbol{A}^T\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 5 \end{bmatrix} \implies \begin{cases} \lambda_1 = 6 \longleftrightarrow \boldsymbol{v}_1 = \frac{1}{\sqrt{30}}(1,2,5)^T \\ \lambda_2 = 1 \longleftrightarrow \boldsymbol{v}_2 = \frac{1}{\sqrt{5}}(-2,1,0)^T \\ \lambda_3 = 0 \longleftrightarrow \boldsymbol{v}_2 = \frac{1}{\sqrt{6}}(-1,-2,1)^T. \end{cases}$$

$$\boldsymbol{A}\boldsymbol{A}^T = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} \implies \begin{cases} \lambda_1 = 6 \longleftrightarrow \boldsymbol{u}_1 = \frac{1}{\sqrt{5}}(2,1)^T \\ \lambda_2 = 1 \longleftrightarrow \boldsymbol{u}_2 = \frac{1}{\sqrt{5}}(1,-2)^T. \end{cases}$$

The singular value decomposition of $\boldsymbol{A}$ is hence given by

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$$

where

$$\boldsymbol{U} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{V} = \frac{1}{\sqrt{30}}\begin{bmatrix} 1 & -2\sqrt{6} & -\sqrt{5} \\ 2 & \sqrt{6} & -2\sqrt{5} \\ 5 & 0 & \sqrt{5} \end{bmatrix}.$$

(b) The pseudoinverse of $\boldsymbol{A}$ is given by

$$\begin{aligned}
\boldsymbol{A}^+ &= \boldsymbol{V}\boldsymbol{\Sigma}^+\boldsymbol{U}^T \\
&= \begin{bmatrix} -1/3 & 5/6 \\ 1/3 & -1/3 \\ 1/3 & 1/6 \end{bmatrix}.
\end{aligned}$$

(c) The projection matrix onto the row space of $\boldsymbol{A}$ is given by

$$\boldsymbol{A}^{+}\boldsymbol{A} = \begin{bmatrix} 5/6 & -1/3 & 1/6 \\ -1/3 & 1/3 & 1/3 \\ 1/6 & 1/3 & 5/6 \end{bmatrix}.$$

(d) Since $\boldsymbol{A}$ has full row rank, there is a right inverse for $\boldsymbol{A}$. We can have

$$\boldsymbol{A}\boldsymbol{A}^{+} = \boldsymbol{I}$$

and hence the pseudoinverse $\boldsymbol{A}^{+}$ obtained in (b) is a right inverse for $\boldsymbol{A}$.

(e) The shortest least squares solution is

$$\boldsymbol{x} = \boldsymbol{A}^{+}\boldsymbol{b} = \begin{bmatrix} -5/6 \\ 4/3 \\ 11/6 \end{bmatrix}.$$