EE 2030 Linear Algebra Chi-chao Chao

Spring 2010

Final Examination

7:00pm to 10:00pm, June 18, 2010

Problems for Solution:

1. (15%) True or false. (If it is true, prove it. Otherwise, find a counterexample or explain why it is false.)
   (a) (5%) If $\boldsymbol{A}$ is a symmetric invertible matrix, then $\boldsymbol{A}^{-1}$ is also symmetric.
   (b) (5%) The following two matrices are similar:
   $$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$
   (c) (5%) The matrix
   $$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix}$$
   is positive definite.

2. (35%) This problem is about the matrices with entries 1, 2, 3, $\ldots$, $n-1$ just above and just below the main diagonal. All other entries are zero:
   $$\boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{A}_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}, \quad \boldsymbol{A}_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$
   $$\boldsymbol{A}_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 4 \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix}, \quad \boldsymbol{A}_6 = \ldots$$
   (a) (5%) Find the complete solution to $\boldsymbol{A}_3\boldsymbol{x} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$.
   (b) (5%) Give a basis for the left nullspace of $\boldsymbol{A}_3$.
   (c) (5%) Find the projection matrix onto the column space of $\boldsymbol{A}_3$.
   (d) (5%) Find the eigenvalues of $\boldsymbol{A}_3$.

(e) (5%) Two eigenvalues of $\boldsymbol{A}_4$ are approximately 3.65 and 0.822. Find the other two eigenvalues using

$$\boldsymbol{M}^{-1}\boldsymbol{A}_4\boldsymbol{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = -\boldsymbol{A}_4.$$

(Hint: If a matrix $\boldsymbol{A}$ is similar to $-\boldsymbol{A}$, what properties should the eigenvalues of $\boldsymbol{A}$ have?)

(f) (5%) Show that $\boldsymbol{A}_5$ is not invertible.

(g) (5%) Is $\boldsymbol{A}_6$ diagonalizable? Why or why not?

3. (15%) Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} 0.3 & c \\ 0.7 & 1-c \end{bmatrix}.$$

(a) (5%) For which value of c is the matrix $\boldsymbol{A}$ not diagonalizable?

(b) (5%) Find the range of the values of c so that $\boldsymbol{A}^n$ approaches a limiting matrix as $n \to \infty$.

(c) (5%) Find $\lim_{n\to\infty} \boldsymbol{A}^n$ (still depending on c) when the limit exists.

4. (15%) This problem consists of three parts:

(a) (5%) If $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ form a basis for $\mathcal{R}^3$, is the matrix with those three columns invertible? Why or why not?

(b) (5%) If $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$, $\boldsymbol{v}_4$ span $\mathcal{R}^3$, give all possible ranks for the matrix with those four columns.

(c) (5%) If $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$ form an orthonormal basis for $\mathcal{R}^3$, and T is the transformation that projects every vector $\boldsymbol{v}$ in $\mathcal{R}^3$ onto the plane spanned by $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$, what is the matrix representation of T in this basis?

5. (20%) Suppose the singular value decomposition $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$ has

$$\boldsymbol{U} = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{V} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

(a) (5%) Find the eigenvalues of $\boldsymbol{A}^T\boldsymbol{A}$.

(b) (5%) Find a basis for the nullspace of $\boldsymbol{A}$.

(c) (5%) Find a basis for the column space of $\boldsymbol{A}$.

(d) (5%) Find a singular value decomposition of $-\boldsymbol{A}^T$.