Solution to Final Examination

1. (a) True.

Since $\boldsymbol{A}$ is symmetric and invertible, we have $\boldsymbol{A}^T = \boldsymbol{A}$ and there exists $\boldsymbol{A}^{-1}$ such that $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$. Therefore, we can have

$$\boldsymbol{I} = \boldsymbol{I}^T = (\boldsymbol{A}\boldsymbol{A}^{-1})^T = (\boldsymbol{A}^{-1})^T\boldsymbol{A}^T = (\boldsymbol{A}^{-1})^T\boldsymbol{A}.$$

Multiple $\boldsymbol{A}^{-1}$ from the right to both sides, and we have

$$\boldsymbol{A}^{-1} = (\boldsymbol{A}^{-1})^T(\boldsymbol{A}\boldsymbol{A}^{-1}) = (\boldsymbol{A}^{-1})^T.$$

That is to say $\boldsymbol{A}^{-1}$ is symmetric.

(b) False.

(i) Let $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and we have

$$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \det\begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^3 = 0.$$

Therefore, the eigenvalues of $\boldsymbol{A}$ are 1, 1, 1.

$$\lambda = 1 \Longrightarrow (\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\boldsymbol{x} = \boldsymbol{0}.$$

Therefore, two independent eigenvectors can be obtained as $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(ii) Let $\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, and we have

$$\det(\boldsymbol{B} - \lambda\boldsymbol{I}) = \det\begin{bmatrix} 1-\lambda & 0 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^3 = 0.$$

The eigenvalues of $\boldsymbol{B}$ are also 1, 1, 1.

$$\lambda = 1 \Longrightarrow (\boldsymbol{B} - \lambda\mathbf{I})\boldsymbol{x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\boldsymbol{x} = \boldsymbol{0}.$$

Therefore, an independent eigenvector can be obtained as $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Since $\boldsymbol{A}$ and $\boldsymbol{B}$ do not have the same number of independent eigenvectors, they are not similar.

(c) True.

Performing elimination, we can have

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix} \Longrightarrow \begin{bmatrix} 2 & 2 & 0 \\ 0 & 3 & 3 \\ 0 & 3 & 8 \end{bmatrix} \Longrightarrow \begin{bmatrix} 2 & 2 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 5 \end{bmatrix} \Longrightarrow \text{pivots: } 2,\, 3,\, 5.$$

The matrix is symmetric and has all pivots>0 (without row exchanges), so it is positive definite.

2. (a) Since by elimination

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 4 \\ 0 & 2 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we know that x_3 is the free variable. Therefore, we can obtain a particular solution as $\boldsymbol{x}_p = (4, 0, 0)^T$ and the nullspace vectors as

$$\boldsymbol{x}_n = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

where $x_3 \in \mathcal{R}$. Finally, the complete solution is given by

$$\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

(b) Since $\boldsymbol{A}_3$ is a symmetric matrix, the left nullspace of $\boldsymbol{A}_3$ is equivalent to the nullspace of $\boldsymbol{A}_3$. A basis can be given as the special solution: $(-2, 0, 1)^T$.

(c) A basis for the column space of $\boldsymbol{A}_3$ can be found as $(0, 1, 0)^T$, $(1, 0, 2)^T$, from which we can have an orthonormal basis given by

$$\boldsymbol{q}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \boldsymbol{q}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

Then the projection matrix can be obtained as

$$\boldsymbol{q}_1\boldsymbol{q}_1^T + \boldsymbol{q}_2\boldsymbol{q}_2^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1/5 & 0 & 2/5 \\ 0 & 1 & 0 \\ 2/5 & 0 & 4/5 \end{bmatrix}.$$

(d) Since

$$\det(\boldsymbol{A_3} - \lambda \boldsymbol{I}) = -\lambda^3 + 5\lambda = 0$$

we can find that the eigenvalues of $\boldsymbol{A_3}$ are 0, $\sqrt{5}$, and $-\sqrt{5}$.

(e) Suppose λ is an eigenvalue of $\boldsymbol{A}_4$, and we can have

$$\begin{aligned} \boldsymbol{A}_4\boldsymbol{v} &= \lambda\boldsymbol{v} \\ \implies (-\boldsymbol{A}_4)\boldsymbol{v} &= (-\lambda)\boldsymbol{v}. \end{aligned}$$

Then $-\lambda$ is an eigenvalue of $-\boldsymbol{A}_4$. Since $\boldsymbol{A}_4$ is similar to $-\boldsymbol{A}_4$, they have the same eigenvalues. Therefore, $-\lambda$ is also an eigenvalue of $\boldsymbol{A}_4$. We can obtain that the other two eigenvalues are -3.65 and -0.822.

(f) Since 0 is an eigenvalue of $\boldsymbol{A}_3$, $\det(\boldsymbol{A}_3) = 0$. Hence, we can have

$$\det(\boldsymbol{A}_5) = -4 \cdot 4 \cdot \det(\boldsymbol{A}_3) = 0.$$

Therefore, $\boldsymbol{A}_5$ is not invertible.

(g) Yes. Since $\boldsymbol{A}_6$ is a real symmetry matrix, it is diagonalizable by Spectral Theorem.

3. (a) We first find the eigenvalues:

$$\begin{aligned} \det(\boldsymbol{A} - \lambda\boldsymbol{I}) &= \begin{vmatrix} 0.3-\lambda & c \\ 0.7 & 1-c-\lambda \end{vmatrix} \\ &= \lambda^2 - (1.3-c)\lambda + (0.3-c). \end{aligned}$$

Solving $\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$, we can have $\lambda = 1, (0.3-c)$. If $1 \neq 0.3 - c$, there will be two independent eigenvectors corresponding to the two distinct eigenvalues and hence the matrix is diagonalizable. We then only have to check the case that $1 = 0.3 - c$, i.e., $c = -0.7$. When $c = -0.7$, $\lambda = 1, 1$, and

$$\boldsymbol{A} - \lambda\boldsymbol{I} = \begin{vmatrix} -0.7 & -0.7 \\ 0.7 & 0.7 \end{vmatrix}.$$

Since $\dim(\mathcal{N}(\boldsymbol{A} - \lambda\boldsymbol{I})) = 1 < 2$, $\boldsymbol{A}$ is not diagonalizable when $c = -0.7$.

(b) From the result in (a), if $c \neq -0.7$, $\boldsymbol{A}$ is diagonalizable and

$$\boldsymbol{A} = \boldsymbol{S} \begin{bmatrix} 1 & 0 \\ 0 & 0.3-c \end{bmatrix} \boldsymbol{S}^{-1}$$

where $\boldsymbol{S}$ is the matrix with two independent eigenvectors as the columns. Therefore,

$$\boldsymbol{A}^n = \boldsymbol{S} \begin{bmatrix} 1 & 0 \\ 0 & (0.3-c)^n \end{bmatrix} \boldsymbol{S}^{-1}.$$

When $-1 < 0.3 - c < 1$, i.e., $-0.7 < c < 1.3$, the limiting matrix exists. It can be easilly checked that the limiting matrix does not exist when $c < -0.7$ or $c \geq 1.3$. The only remaining case we need to check is when $\boldsymbol{A}$ is not diagonalizable, i.e., $c = -0.7$. When $c = -0.7$, $\lambda = 1, 1$, and

$$\boldsymbol{A} - \lambda\boldsymbol{I} = \begin{bmatrix} -0.7 & -0.7 \\ 0.7 & 0.7 \end{bmatrix}.$$

Let the corresponding eigenvector be $(-1/\sqrt{2}\ ,\ 1/\sqrt{2})^T$. Using the Gram-Schmidt method, we can obtain an orthonormal basis as $\boldsymbol{w}_1 = (-1/\sqrt{2}\ , 1/\sqrt{2})^T$,

$\boldsymbol{w}_2 = (1/\sqrt{2}\ ,\ 1/\sqrt{2})^T$ for $\mathcal{R}^2$. For any vector $\boldsymbol{x} \in \mathcal{R}^2$, we can have $\boldsymbol{x} = a_1\boldsymbol{w}_1 + a_2\boldsymbol{w}_2$, and

$$\boldsymbol{A}\boldsymbol{x} = A(a_1\boldsymbol{w}_1 + a_2\boldsymbol{w}_2) = a_1\boldsymbol{w}_1 + a_2\boldsymbol{A}\boldsymbol{w}_2.$$

It can be checked that $A\boldsymbol{w_2} = 1.4\boldsymbol{w_1} + \boldsymbol{w_2}$, and thus

$$\boldsymbol{A}\boldsymbol{x} = a_1\boldsymbol{w}_1 + a_2(1.4\boldsymbol{w}_1 + \boldsymbol{w}_2) = (a_1 + 1.4a_2)\boldsymbol{w}_1 + a_2\boldsymbol{w}_2$$

which leads to

$$\boldsymbol{A}^n\boldsymbol{x} = (a_1 + 1.4na_2)\boldsymbol{w}_1 + a_2\boldsymbol{w}_2.$$

Therefore, as $n \to \infty$ the limit of $\boldsymbol{A}^n\boldsymbol{x}$ does not exist if $a_2 \neq 0$. We can have

$$\boldsymbol{A}^n = \boldsymbol{A}^{n-1}\begin{bmatrix}\boldsymbol{x}_1 & \boldsymbol{x}_2\end{bmatrix}$$

where $\boldsymbol{x}_1 = (0.3,\ 0.7)^T$ and $\boldsymbol{x}_2 = (-0.7,\ 1.7)^T$. Then the limit of $\boldsymbol{A}^n$ does not exist as $n \to \infty$ because $\boldsymbol{a_1}, \boldsymbol{a_2}$ are not both multiples of $\boldsymbol{w_1}$. Consequently, the limiting matrix exists only when $-0.7 < c < 1.3$.

(c) From the result in (b), $\boldsymbol{A}$ is diagonalizable when the limiting matrix exists. Since $\boldsymbol{A}$ has eigenvalues $\lambda = 1, (0.3 - c)$, we first find the corresponding eigenvectors. When $\lambda = 1$,

$$\boldsymbol{A} - \lambda\boldsymbol{I} = \begin{bmatrix}-0.7 & c\\ 0.7 & -c\end{bmatrix}.$$

Let the corresponding eigenvector be $\boldsymbol{s_1} = (c,\ 0.7)^T$. When $\lambda = 0.3 - c$,

$$\boldsymbol{A} - \lambda\boldsymbol{I} = \begin{bmatrix}c & c\\ 0.7 & 0.7\end{bmatrix}.$$

Let the corresponding eigenvector be $\boldsymbol{s_2} = (1,\ -1)^T$. Let

$$\boldsymbol{S} = \begin{bmatrix}\boldsymbol{s_1} & \boldsymbol{s_2}\end{bmatrix} = \begin{bmatrix}c & 1\\ 0.7 & -1\end{bmatrix}$$

and we can obtain

$$\boldsymbol{S}^{-1} = \frac{1}{c + 0.7}\begin{bmatrix}1 & 1\\ 0.7 & -c\end{bmatrix}.$$

We then have

$$\boldsymbol{A} = \boldsymbol{S}\begin{bmatrix}1 & 0\\ 0 & (0.3 - c)\end{bmatrix}\boldsymbol{S}^{-1}.$$

Therefore,

$$\boldsymbol{A}^n = \boldsymbol{S}\begin{bmatrix}1^n & 0\\ 0 & (0.3 - c)^n\end{bmatrix}\boldsymbol{S}^{-1}.$$

When $-1 < 0.3 - c < 1$, we can have

$$\begin{aligned}\lim_{n\to\infty}\boldsymbol{A}^n &= \boldsymbol{S}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\boldsymbol{S}^{-1}\\ &= \begin{bmatrix}c & 1\\ 0.7 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\frac{1}{c + 0.7}\begin{bmatrix}1 & 1\\ 0.7 & -c\end{bmatrix}\\ &= \frac{1}{c + 0.7}\begin{bmatrix}c & c\\ 0.7 & 0.7\end{bmatrix}.\end{aligned}$$

4. (a) Let

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} \end{bmatrix}.$$

Since $\boldsymbol{v_1}$, $\boldsymbol{v_2}$, $\boldsymbol{v_3}$ are independent, we know that $\text{rank}(\boldsymbol{A}) = 3$ and $\boldsymbol{A}$ is full-rank. Therefore, $\boldsymbol{A}$ is invertible.

(b) Let

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \boldsymbol{v_4} \end{bmatrix}.$$

Since $\boldsymbol{v_1}$, $\boldsymbol{v_2}$, $\boldsymbol{v_3}$, $\boldsymbol{v_4}$ span $\mathcal{R}^3$, we know that $\text{rank}(\boldsymbol{A}) = \dim(\mathcal{C}(\boldsymbol{A})) = 3$.

(c) Since T is the transformation that projects onto the plane spanned by $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$, we can have

$$\begin{aligned} T(\boldsymbol{q}_1) &= \boldsymbol{q}_1 = 1 \cdot \boldsymbol{q}_1 + 0 \cdot \boldsymbol{q}_2 + 0 \cdot \boldsymbol{q}_3 \\ T(\boldsymbol{q}_2) &= \boldsymbol{q}_2 = 0 \cdot \boldsymbol{q}_1 + 1 \cdot \boldsymbol{q}_2 + 0 \cdot \boldsymbol{q}_3 \\ T(\boldsymbol{q}_3) &= \boldsymbol{0} = 0 \cdot \boldsymbol{q}_1 + 0 \cdot \boldsymbol{q}_2 + 0 \cdot \boldsymbol{q}_3. \end{aligned}$$

Therefore, the matrix representation of T in this basis is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

5. $\boldsymbol{A} = \boldsymbol{U\Sigma V}^T$ is a 3×4 matrix where

$$\boldsymbol{U} = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}, \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \boldsymbol{V} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

(a) Since $\boldsymbol{A}^T\boldsymbol{A}$ is 4×4, it has 4 eigenvalues. The nonzero eigenvalues of $\boldsymbol{A}^T\boldsymbol{A}$ are the squares of the singular values of $\boldsymbol{A}$, which are given by $1^2 = 1$ and $4^2 = 16$. Therefore, the eigenvalues of $\boldsymbol{A}^T\boldsymbol{A}$ are 1, 16, 0, 0.

(b) Since there are 2 nonzero singular values of $\boldsymbol{A}$, the rank of $\boldsymbol{A}$ is 2. Therefore, the dimension of the nullspace of $\boldsymbol{A} = 4 - 2 = 2$. A basis for the nullspace of $\boldsymbol{A}$ can be obtained as the last two columns of $\boldsymbol{V}$, i.e.,

$$\begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

(c) Since the dimension of the column space of $\boldsymbol{A}$ is 2, a basis for the column space of $\boldsymbol{A}$ can be obtained as the first two columns of $\boldsymbol{U}$, i.e.,

$$\begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}.$$

(d) A singular value decomposition of $-\boldsymbol{A}^T$ can be given by

$$-\boldsymbol{A}^T = -(\boldsymbol{U\Sigma V}^T)^T = -\boldsymbol{V\Sigma}^T\boldsymbol{U}^T = \boldsymbol{U}'\boldsymbol{\Sigma}'\boldsymbol{V}'^T$$

where $\boldsymbol{U}' = -\boldsymbol{V}$, $\boldsymbol{\Sigma}' = \boldsymbol{\Sigma}^T$ and $\boldsymbol{V}' = \boldsymbol{U}$.