EE 2030 Linear Algebra Chi-chao Chao
Spring 2010

**Midterm Examination No. 1**
7:00pm to 10:00pm, March 26, 2010

**Problems for Solution:**

1. (15%) Your classmate Emily performed the usual elimination steps to convert $\boldsymbol{A}$ to $\boldsymbol{U}$, obtaining

$$\boldsymbol{U} = \begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & 2 & 2 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   (a) (8%) Find all the vectors in the nullspace $\mathcal{N}(\boldsymbol{A})$.

   (b) (7%) Emily gave you a matrix

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}$$

   and told you that $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$. If $\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$, then $\boldsymbol{U}\boldsymbol{x} = \boldsymbol{c}$. Find $\boldsymbol{c}$.

2. (a) (5%) Find the inverse of

$$\boldsymbol{A}_4 = \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

   (b) (5%) First guess the inverse of

$$\boldsymbol{A}_5 = \begin{bmatrix} 1 & -a & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 \\ 0 & 0 & 1 & -c & 0 \\ 0 & 0 & 0 & 1 & -d \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

   Then multiply to confirm.

3. (10%) Find the $\boldsymbol{P}\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$ factorization for

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 3 & 4 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 0 & 0 \end{bmatrix}$$

   where $\boldsymbol{P}$ is a permutation matrix, $\boldsymbol{L}$ is a lower triangular matrix with unit diagonal, and $\boldsymbol{U}$ is an upper triangular matrix.

4. (15%) If the following statement is true, prove it; otherwise, find a counterexample. Recall that $\boldsymbol{M}$ is the vector space of all real $2 \times 2$ matrices.

   (a) (8%) The invertible matrices in $\boldsymbol{M}$ form a subspace.

   (b) (7%) The matrices with the sum of the components in each row equal to zero in $\boldsymbol{M}$ form a subspace.

5. (a) (6%) Find column vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ so that $\boldsymbol{A} = \boldsymbol{u}\boldsymbol{v}^T$:

$$\boldsymbol{A} = \begin{bmatrix} 2 & 4 & 1 & 3 \\ -4 & -8 & -2 & -6 \\ 6 & 12 & 3 & 9 \end{bmatrix}.$$

   (b) (4%) Find the rank of $\boldsymbol{A}$.

6. (10%) Under what condition on $b_1$, $b_2$, $b_3$ is this system solvable? Find all solutions when that condition holds:

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 4 & 9 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

7. (15%) Find matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ with the given property or explain why you cannot:

   (a) (8%) The only solution to $\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

   (b) (7%) The only solution to $\boldsymbol{B}\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

8. (15%) The matrix $\boldsymbol{A}$ has its nullspace $\mathcal{N}(\boldsymbol{A})$ spanned by the following three vectors:

$$\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ -1 \\ 3 \\ 1 \end{bmatrix}.$$

   (a) (7%) Give a matrix $\boldsymbol{B}$ such that its column space $\mathcal{C}(\boldsymbol{B})$ is the same as $\mathcal{N}(\boldsymbol{A})$.

   (b) (8%) For some vector $\boldsymbol{b}$, you are told that a particular solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is

$$\boldsymbol{x}_p = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Now, your classmate Catherine tells you that a second solution is:

$$\boldsymbol{x}_C = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}$$

while your other classmate Jonathan tells you "No, Catherine's solution cannot be right, but here is a second solution that is correct:"

$$\boldsymbol{x}_J = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}.$$

Is Catherine's solution correct, or Jonathan's solution, or are both correct?