Solution to Midterm Examination No. 1

1. (a) Since $\boldsymbol{U}$ is obtained form $\boldsymbol{A}$ through elimination, we know that $\mathcal{N}(\boldsymbol{A}) = \mathcal{N}(\boldsymbol{U})$. Transform $\boldsymbol{U}$ into the reduced row echelon (RRE) form:

$$\begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & 2 & 2 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & -5 & 15 \\ 0 & 2 & 2 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ (subtract } 2 \times \text{row } 2\text{)}$$

$$\Longrightarrow \begin{bmatrix} 1 & 0 & -5 & 15 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ (divide by 2)}$$

The pivot variables are x_1 and x_2, and the free variables are x_3 and x_4.

- Given $(x_3, x_4) = (1, 0)$, we have $(x_1, x_2) = (5, -1)$.
- Given $(x_3, x_4) = (0, 1)$, we have $(x_1, x_2) = (-15, 3)$.

Therefore, we have

$$\mathcal{N}(\boldsymbol{A}) = \mathcal{N}(\boldsymbol{U}) = \left\{ x_3 \begin{bmatrix} 5 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -15 \\ 3 \\ 0 \\ 1 \end{bmatrix} : \ x_3, x_4 \in \mathcal{R} \right\}.$$

(b) We have

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}$$

which records the elimination steps. Hence

$$\begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix} \text{ (subtract } 2 \times \text{row 1)}$$

$$\Longrightarrow \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix} \text{ (subtract } -1 \times \text{row 1)}$$

$$\Longrightarrow \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \text{ (subtract } 3 \times \text{row 2)}$$

Therefore,

$$\boldsymbol{c} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}.$$

2. (a) Perform Gauss-Jordan method to find $\boldsymbol{A}_4^{-1}$ as follows:

$$
\begin{aligned}
&\left[\begin{array}{c|c} \boldsymbol{A}_4 & \boldsymbol{I} \end{array}\right] \\
= &\left[\begin{array}{cccc|cccc} 1 & -a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -c & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \\
\Longrightarrow &\left[\begin{array}{cccc|cccc} 1 & -a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad (\text{subtract } -c \times \text{row } 4\ ) \\
\Longrightarrow &\left[\begin{array}{cccc|cccc} 1 & -a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & b & bc \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad (\text{subtract } -b \times \text{row } 3\ ) \\
\Longrightarrow &\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & a & ab & abc \\ 0 & 1 & 0 & 0 & 0 & 1 & b & bc \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad (\text{subtract } -a \times \text{row } 2\ )
\end{aligned}
$$

We can hence obtain

$$
\boldsymbol{A}_4^{-1} = \begin{bmatrix} 1 & a & ab & abc \\ 0 & 1 & b & bc \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

(b) Form the result of (a), we guess that the inverse of $\boldsymbol{A}_5$ is given by

$$
\begin{bmatrix} 1 & a & ab & abc & abcd \\ 0 & 1 & b & bc & bcd \\ 0 & 0 & 1 & c & cd \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
$$

Since

$$
\begin{aligned}
\begin{bmatrix} 1 & a & ab & abc & abcd \\ 0 & 1 & b & bc & bcd \\ 0 & 0 & 1 & c & cd \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{A}_5 &= \begin{bmatrix} 1 & a & ab & abc & abcd \\ 0 & 1 & b & bc & bcd \\ 0 & 0 & 1 & c & cd \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 \\ 0 & 0 & 1 & -c & 0 \\ 0 & 0 & 0 & 1 & -d \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \boldsymbol{I}
\end{aligned}
$$

and

$$\boldsymbol{A}_5 \begin{bmatrix} 1 & a & ab & abc & abcd \\ 0 & 1 & b & bc & bcd \\ 0 & 0 & 1 & c & cd \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -a & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 \\ 0 & 0 & 1 & -c & 0 \\ 0 & 0 & 0 & 1 & -d \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & ab & abc & abcd \\ 0 & 1 & b & bc & bcd \\ 0 & 0 & 1 & c & cd \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \boldsymbol{I}$$

we have confirmed

$$\boldsymbol{A}_5^{-1} = \begin{bmatrix} 1 & a & ab & abc & abcd \\ 0 & 1 & b & bc & bcd \\ 0 & 0 & 1 & c & cd \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. Exchanging rows 1 and 4, we can have

$$\begin{bmatrix} 0 & 0 & 3 & 4 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

Then elimination gives

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix} \quad \text{(subtract } 2 \times \text{row 1)}$$

$$\Longrightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix} \quad \text{(subtract } -1 \times \text{row 2)}$$

$$\Longrightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(subtract } 1 \times \text{row 3)}$$

Therefore, we can obtain

$$\boldsymbol{PA} = \boldsymbol{LU}$$

where

$$\boldsymbol{P} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 3 & 4 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\boldsymbol{U} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

4. (a) The invertible matrices in $\boldsymbol{M}$ do not form a subspace. A counterexample is given as follows. Consider an invertible matrix $\boldsymbol{A}$ in $\boldsymbol{M}$ and $c = 0$, and we have

$$c\boldsymbol{A} = 0\boldsymbol{A} = \boldsymbol{0}$$

which is not invertible in $\boldsymbol{M}$. Therefore, the invertible matrices in $\boldsymbol{M}$ do not form a subspace.

(b) The matrices with the sum of the components in each row equal to zero in $\boldsymbol{M}$ form a subspace. Consider

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } \boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

where $a_{11} + a_{12} = a_{21} + a_{22} = b_{11} + b_{12} = b_{21} + b_{22} = 0$. We need to check the following two conditions.

- For

$$\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

we have

$$a_{11} + b_{11} + a_{12} + b_{12} = (a_{11} + a_{12}) + (b_{11} + b_{12}) = 0 + 0 = 0$$

and

$$a_{21} + b_{21} + a_{22} + b_{22} = (a_{12} + a_{22}) + (b_{12} + b_{22}) = 0 + 0 = 0.$$

Therefore, $\boldsymbol{A} + \boldsymbol{B}$ is still a matrix with the sum of the components in each row equal to zero in $\boldsymbol{M}$.

- For any c, consider

$$c\boldsymbol{A} = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}.$$

Since we have

$$ca_{11} + ca_{12} = c(a_{11} + a_{12}) = 0$$

and

$$ca_{21} + ca_{22} = c(a_{21} + a_{22}) = 0$$

$c\boldsymbol{A}$ is still a matrix with the sum of the components in each row equal to zero in $\boldsymbol{M}$.

Since the above two conditions are satisfied, matrices with the sum of the components in each row equal to zero in $\boldsymbol{M}$ form a subspace.

5. (a) Let

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}.$$

We can have

$$\boldsymbol{A} = \boldsymbol{u}\boldsymbol{v}^T = \begin{bmatrix} u_1\boldsymbol{v}^T \\ u_2\boldsymbol{v}^T \\ u_3\boldsymbol{v}^T \end{bmatrix} = \begin{bmatrix} 2 & 4 & 1 & 3 \\ -4 & -8 & -2 & -6 \\ 6 & 12 & 3 & 9 \end{bmatrix}$$

yielding

$$\begin{aligned} \boldsymbol{v}^T &= \frac{1}{u_1}\begin{bmatrix} 2 & 4 & 1 & 3 \end{bmatrix} \\ u_2 &= -2u_1 \\ u_3 &= 3u_1. \end{aligned}$$

Taking $u_1 = a$ for some $a \neq 0$, we can obtain

$$\boldsymbol{u} = a\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \text{ and } \boldsymbol{v} = \frac{1}{a}\begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix}.$$

For example, letting $a = 1$, we can have

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix}.$$

(b) Since $\boldsymbol{A} = \boldsymbol{u}\boldsymbol{v}^T$, the rank of $\boldsymbol{A}$ is 1.

6. We can solve this system by the following procedure:

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 4 & 9 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & -2b_1 + b_2 \\ 4 & 9 & -8 & b_3 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & -2b_1 + b_2 \\ 0 & 1 & 0 & -4b_1 + b_3 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & -2b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 - b_2 + b_3 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 5b_1 - 2b_2 \\ 0 & 1 & 0 & -2b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 - b_2 + b_3 \end{array}\right]$$

It can be seen that the system is solvable only if $-2b_1 - b_2 + b_3 = 0$, i.e., $b_3 = 2b_1 + b_2$. If $b_3 = 2b_1 + b_2$, we can go on solving

$$\begin{cases} x_1 \quad -2x_3 = 5b_1 - 2b_2 \\ \quad x_2 \quad = -2b_1 + b_2. \end{cases}$$

Letting $x_3 = 0$, we can find a particular solution $\boldsymbol{x}_p = \begin{bmatrix} 5b_1 - 2b_2 \\ -2b_1 + b_2 \\ 0 \end{bmatrix}$. And the general solution to the homogeneous system is $\boldsymbol{x}_n = x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. Therefore, we can obtain the complete solution

$$\boldsymbol{x} = x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 5b_1 - 2b_2 \\ -2b_1 + b_2 \\ 0 \end{bmatrix}$$

where $x_3 \in \boldsymbol{R}$.

7. (a) It is clear that $\boldsymbol{A}$ is 3×2. For $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to be the only solution to $\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, the nullspace of $\boldsymbol{A}$ must contain the zero vector only. Hence, the rank of $\boldsymbol{A}$ (the number of pivots) should be 2. Let $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix}$, where $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are column vectors. We have

$$\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

which gives

$$\boldsymbol{a}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

And $\boldsymbol{a}_2$ can be any 3×1 column vector which is not a multiple of $\boldsymbol{a}_1$.

(b) It is clear that $\boldsymbol{B}$ is 2×3. For $\boldsymbol{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ to be the only one solution to $\boldsymbol{Bx} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the nullspace of $\boldsymbol{B}$ must contain the zero vector only. Hence, the rank of $\boldsymbol{B}$ should be 3. Yet as the number of rows of $\boldsymbol{B}$ is only 2, the rank of $\boldsymbol{B}$ cannot be 3. Therefore, $\boldsymbol{B}$ does not exist.

8. (a) $\boldsymbol{B}$ can be chosen as

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ -1 & 1 & 3 \\ 3 & 4 & 1 \end{bmatrix}$$

or any 4×3 real matrix whose columns are independent linear combinations of columns of the above matrix.

(b) This problem is equivalent to finding c such that

$$\boldsymbol{x}' = \begin{bmatrix} 1 \\ 1 \\ 3 \\ c \end{bmatrix}$$

is a solution to $\boldsymbol{Ax} = \boldsymbol{b}$. Since $\boldsymbol{x}_p$ is also a solution to $\boldsymbol{Ax} = \boldsymbol{b}$, we can obtain that

$$\boldsymbol{x}'' = \boldsymbol{x}' - \boldsymbol{x}_p = \begin{bmatrix} 1 \\ 1 \\ 3 \\ c \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ c-4 \end{bmatrix}$$

must be in the nullspace $\mathcal{N}(\boldsymbol{A})$. That is, the system

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ -1 & 1 & 3 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ c-4 \end{bmatrix}$$

should be solvable. By performing elimination, we can have

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 2 & 1 & -1 & -1 \\ -1 & 1 & 3 & 0 \\ 3 & 4 & 1 & c-4 \end{array}\right]$$
$$\Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & 4 & 4 & c-4 \end{array}\right]$$
$$\Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & c \end{array}\right]$$

For this system to be solvable, we must have $c = 0$. Therefore, Catherine's solution

$$\boldsymbol{x}_C = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}$$

is correct while Jonathan's is not.