EE 2030 Linear Algebra Chi-chao Chao
Spring 2010

Midterm Examination No. 2
7:00pm to 10:00pm, May 7, 2010

Problems for Solution:

1. (15%) Find a basis for each of the four subspaces for

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

2. (10%) Suppose $\boldsymbol{A}$ is a 5 by 4 matrix with rank 4.
   (a) (5%) Show that $\boldsymbol{Ax} = \boldsymbol{b}$ has no solution when the 5 by 5 matrix $[\,\boldsymbol{A}\ \boldsymbol{b}\,]$ is invertible.
   (b) (5%) Show that $\boldsymbol{Ax} = \boldsymbol{b}$ is solvable when $[\,\boldsymbol{A}\ \boldsymbol{b}\,]$ is singular.

3. (10%) Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}.$$

   Given the vector

$$\boldsymbol{b} = \begin{bmatrix} 13 \\ 27 \end{bmatrix}$$

   in the column space of $\boldsymbol{A}$, find a vector $\boldsymbol{x}_r$ in the row space of $\boldsymbol{A}$ such that

$$\boldsymbol{A}\boldsymbol{x}_r = \boldsymbol{b}.$$

4. (10%) Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

   Suppose $\boldsymbol{P}_1$ is the projection matrix onto the the one-dimensional subspace spanned by the first column of $\boldsymbol{A}$. Suppose $\boldsymbol{P}_2$ is the projection matrix onto the two-dimensional column space of $\boldsymbol{A}$. Compute the product $\boldsymbol{P}_2\boldsymbol{P}_1$. (Think before you compute.)

5. (10%) You are told that the least-square linear fit to three points $(0, b_1)$, $(1, b_2)$, and $(2, b_3)$ is $C + Dt$ for $C = 1$ and $D = -2$. That is, the fit is $1 - 2t$. In this problem, you will work backwards from this fit to reason about the unknown values $\boldsymbol{b} = (b_1\ b_2\ b_3)^T$ at $t = 0, 1, 2$.

(a) (5%) Find the explicit equations that b_1, b_2, b_3 must satisfy for $1 - 2t$ to be the least-square linear fit. (You should simplify your equations as much as possible.)

(b) (5%) If all the three points fall exactly on the line $1 - 2t$, find $\boldsymbol{b}$. Check that this satisfies your equations in (a).

6. (20%) The matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 & -7 \\ 2 & 4 & 1 & -5 \\ 1 & 2 & 2 & -16 \end{bmatrix}$$

is converted into the reduced row echelon form by the usual elimination steps, resulting in the matrix:

$$\boldsymbol{R} = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) (5%) What is the maximum number of columns of $\boldsymbol{A}$ that form an independent set of vectors? (You should explain your result.)

(b) (5%) Give an orthonormal basis for the row space of $\boldsymbol{A}$.

(c) (5%) Given the vector $\boldsymbol{b} = (2 \;\; 5 \;\; -9 \;\; 3)^T$, find the closest vector $\boldsymbol{p}$ to $\boldsymbol{b}$ in the row space of $\boldsymbol{A}$.

(d) (5%) Given the vector $\boldsymbol{b} = (2 \;\; 5 \;\; -9 \;\; 3)^T$, find the closest vector $\boldsymbol{p}'$ to $\boldsymbol{b}$ in the nullspace of $\boldsymbol{A}$.

7. (10%) This problem shows in two ways that $\det \boldsymbol{A} = 0$ (the x's are any numbers):

$$\boldsymbol{A} = \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}.$$

(a) (5%) Show that the columns are linearly dependent.

(b) (5%) Explain why all the terms are zero in the big formula for $\det \boldsymbol{A}$.

8. (15%) Let

$$\boldsymbol{A}_n = \begin{bmatrix} a_1 & -1 & 0 & 0 & \cdots & 0 \\ 1 & a_2 & -1 & 0 & \cdots & 0 \\ 0 & 1 & a_3 & -1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a_{n-1} & -1 \\ 0 & 0 & \cdots & 0 & 1 & a_n \end{bmatrix}.$$

(a) (7%) Show for $n \geq 3$ that $\det \boldsymbol{A}_n = a_n \det \boldsymbol{A}_{n-1} + \det \boldsymbol{A}_{n-2}$.

(b) (8%) Calculate $\det \boldsymbol{A}_6$ for the cases that (i) $a_j = j$, for $j = 1, 2, \ldots, 6$, and (ii) $a_j = 6 - j$, for $j = 1, 2, \ldots, 6$.