Solution to Midterm Examination No. 2

1. Perform Gaussian elimination to obtain the reduced row echelon form $\boldsymbol{R}$ of $\boldsymbol{A}$ as follows:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \boldsymbol{R}.$$

- Row space:

  The row space of $\boldsymbol{A}$ is the same as that of $\boldsymbol{R}$. We have a basis of the row space of $\boldsymbol{A}$ as

  $$(0\ 1\ 2\ 0\ \ -2), (0\ 0\ 0\ 1\ 2).$$

- Column space:

  It is obvious that the second and forth columns of $\boldsymbol{R}$ form a basis for the column space of $\boldsymbol{R}$. Hence the second and forth columns of $\boldsymbol{A}$ form a basis for the column space of $\boldsymbol{A}$. Therefore, a basis for the column space of $\boldsymbol{A}$ is

  $$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}.$$

- Nullspace:

  By $\boldsymbol{R}$, we have the special solutions

  $$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

  which form a basis for $\mathcal{N}(\boldsymbol{A})$.

- Left nullspace:

  To find the left nullspace of $\boldsymbol{A}$, we perform Gaussian elimination to obtain the reduced row echelon form of $\boldsymbol{A}^T$:

  $$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 4 & 1 \\ 4 & 6 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

  $$\Longrightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then we have the special solution

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

which forms a basis for $\mathcal{N}(\boldsymbol{A}^{\mathcal{T}})$.

2. (a) Since the 5×5 matrix $\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \end{bmatrix}$ is invertible, there are 5 nonzero pivots and all the columns are independent. Then $\boldsymbol{b}$ cannot be a linear combination of the columns of $\boldsymbol{A}$. Therefore, $\boldsymbol{Ax} = \boldsymbol{b}$ has no solution.

(b) We know that $\boldsymbol{A}$ is a 5×4 matrix with rank 4. That is to say $\boldsymbol{A}$ is with full column rank. Since $\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \end{bmatrix}$ is singular and $\boldsymbol{A}$ is with full column rank, there is only one free variable in the column of $\boldsymbol{b}$. Then $\boldsymbol{b}$ can be a linear combination of the columns of $\boldsymbol{A}$. Therefore, $\boldsymbol{Ax} = \boldsymbol{b}$ is solvable.

3. Since $\boldsymbol{x}_r$ is in the row space of $\boldsymbol{A}$, we have

$$\boldsymbol{x}_r = \boldsymbol{A}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \text{ for some } x_1,\, x_2 \in \mathcal{R}.$$

Then

$$\boldsymbol{A}\boldsymbol{x}_r = \boldsymbol{A}\boldsymbol{A}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{b}$$

where

$$\boldsymbol{A}\boldsymbol{A}^T = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ 9 & 18 \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} 13 \\ 27 \end{bmatrix}.$$

We can solve x_1 and x_2 by

$$\begin{bmatrix} 5 & 9 \\ 9 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 13 \\ 27 \end{bmatrix} \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Hence,

$$\boldsymbol{x}_r = \boldsymbol{A}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}.$$

4. Let $\boldsymbol{K_1}$ be the subspace spanned by the first column of $\boldsymbol{A}$ and $\boldsymbol{K_2}$ be the column space of $\boldsymbol{A}$. Obviously, $\boldsymbol{K_1}$ is contained in $\boldsymbol{K_2}$. For any vector $\boldsymbol{x}$, the projection vector of $\boldsymbol{x}$ onto $\boldsymbol{K_1}$ is

$$\boldsymbol{P_1x} \in \boldsymbol{K_1} \subset \boldsymbol{K_2}.$$

Since $\boldsymbol{P_1x} \in \boldsymbol{K_2}$, the projection vector of $\boldsymbol{P_1x}$ onto $\boldsymbol{K_2}$ is $\boldsymbol{P_1x}$ itself. Thus,

$$\boldsymbol{P_2P_1x} = \boldsymbol{P_2}(\boldsymbol{P_1x}) = \boldsymbol{P_1x}, \quad \text{for any vector } \boldsymbol{x}.$$

Therefore,

$$P_2P_1 = P_1 = \frac{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 \end{bmatrix}}{\begin{bmatrix} 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}} = \frac{1}{6} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}.$$

5. (a) Consider the least-square linear fit to $(0, b_1)$, $(1, b_2)$, and $(2, b_3)$, and let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \; \widehat{\boldsymbol{x}} = \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \text{ and } \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Since

$$\boldsymbol{A}^T \boldsymbol{A} \widehat{\boldsymbol{x}} = \boldsymbol{A}^T \boldsymbol{b}$$

we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} -3 \\ -7 \end{bmatrix} = \begin{bmatrix} b_1 + b_2 + b_3 \\ b_2 + 2b_3 \end{bmatrix}.$$

The equations that b_1, b_2, and b_3 must satisfy are

$$\begin{cases} b_1 + b_2 + b_3 = -3 \\ b_2 + 2b_3 = -7. \end{cases}$$

(b) Since all the three points $(0, b_1)$, $(1, b_2)$, and $(2, b_3)$ fall on the line $1 - 2t$, we can have

$$\begin{aligned} b_1 &= 1 - 2 \times 0 = 1 \\ b_2 &= 1 - 2 \times 1 = -1 \\ b_3 &= 1 - 2 \times 2 = -3. \end{aligned}$$

That is to say

$$\boldsymbol{b} = (b_1 \; b_2 \; b_3)^T = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}.$$

Then we can have

$$\begin{cases} b_1 + b_2 + b_3 = 1 + (-1) + -3 = -3 \\ b_2 + 2b_3 = -1 + 2 \times (-3) = -7 \end{cases}$$

which satisfy the equations in (a).

6. (a) Since rank($\boldsymbol{A}$) = rank($\boldsymbol{R}$) = 2, the maximum number of columns of $\boldsymbol{A}$ that form an independent set of vectors is equal to 2.

(b) Since the row space of $\boldsymbol{A}$ is equal to that of $\boldsymbol{R}$, take

$$\boldsymbol{a}_1 = \begin{bmatrix}1\\2\\0\\2\end{bmatrix} \text{ and } \boldsymbol{a}_2 = \begin{bmatrix}0\\0\\1\\-9\end{bmatrix}$$

as a basis for the row space of $\boldsymbol{A}$. Then perform the Gram-Schmidt process as follows.

1. Take $\boldsymbol{A}_1 = \boldsymbol{a}_1$. Then $\boldsymbol{q}_1 = \frac{\boldsymbol{A}_1}{||\boldsymbol{A}_1||} = \frac{1}{3}\begin{bmatrix}1\\2\\0\\2\end{bmatrix}$.

2. Take $\boldsymbol{A}_2 = \boldsymbol{a}_2 - (\boldsymbol{q}_1^T\boldsymbol{a}_2)\boldsymbol{q}_1 = \begin{bmatrix}2\\4\\1\\-5\end{bmatrix}$. Then $\boldsymbol{q}_2 = \frac{\boldsymbol{A}_2}{||\boldsymbol{A}_2||} = \frac{1}{\sqrt{46}}\begin{bmatrix}2\\4\\1\\-5\end{bmatrix}$.

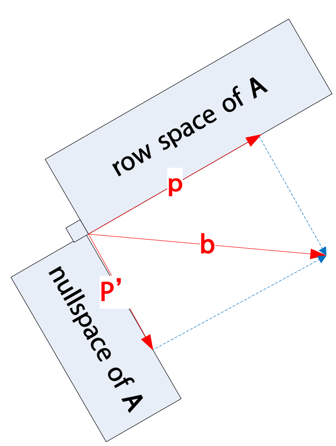
Therefore, we obtain an orthonormal basis for the row space of $\boldsymbol{A}$ as

$$\frac{1}{3}\begin{bmatrix}1\\2\\0\\2\end{bmatrix} \text{ and } \frac{1}{\sqrt{46}}\begin{bmatrix}2\\4\\1\\-5\end{bmatrix}.$$

(c) This is equivalent to finding the projection of $\boldsymbol{b}$ onto the row space of $\boldsymbol{A}$. We can obtain

$$\boldsymbol{p} = (\boldsymbol{q}_1^T\boldsymbol{b})\boldsymbol{q}_1 + (\boldsymbol{q}_2^T\boldsymbol{b})\boldsymbol{q}_2 = \begin{bmatrix}2\\4\\0\\4\end{bmatrix}.$$

(d) By the following figure,



we can have

$$\boldsymbol{p}' = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} 0 \\ 1 \\ -9 \\ -1 \end{bmatrix}.$$

7. (a) Consider the first three columns:

$$\begin{bmatrix} x & x & x \\ x & x & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The rank of this submatrix is at most 2. Therefore, the columns are dependent.

(b) The big formula states that the determinant of $\boldsymbol{A}$ is the sum of 5! simple determinants, times 1 or -1, and every simple determinant chooses one entry from each row and column. If some simple determinant of $\boldsymbol{A}$ avoids all the zero entries in $\boldsymbol{A}$, then it cannot choose one entry from each column. Thus every simple determinant of $\boldsymbol{A}$ must choose at least one zero entry, and hence all the terms are zero in the big formula for det$\boldsymbol{A}$.

8. (a) Let $|\boldsymbol{A_n}| = \det \boldsymbol{A}_n$. First observe that, for $n \geq 3$,

$$|\boldsymbol{A_n}| = \begin{vmatrix} & & & & 0 \\ & \boldsymbol{A}_{n-1} & & & \vdots \\ & & & & 0 \\ & & & & -1 \\ 0 & \cdots & 0 & 1 & a_n \end{vmatrix} = \begin{vmatrix} & & & & 0 & 0 \\ & \boldsymbol{A}_{n-2} & & & \vdots & \vdots \\ & & & & -1 & 0 \\ 0 & \cdots & 0 & 1 & a_{n-1} & -1 \\ 0 & \cdots & 0 & 0 & 1 & a_n \end{vmatrix}.$$

Applying the cofactor formula to the last row, we can have

$$|\boldsymbol{A_n}| = a_n \cdot (-1)^{n+n} |\boldsymbol{A_{n-1}}| + 1 \cdot (-1)^{n+(n-1)} \begin{vmatrix} & & & & 0 \\ & \boldsymbol{A}_{n-2} & & & \vdots \\ & & & & 0 \\ 0 & \cdots & 0 & 1 & -1 \end{vmatrix}$$

$$= a_n|\boldsymbol{A_{n-1}}| - (-1) \cdot (-1)^{(n-1)+(n-1)} |\boldsymbol{A_{n-2}}| \text{ (expansion along the last column)}$$

$$= a_n|\boldsymbol{A_{n-1}}| + |\boldsymbol{A_{n-2}}|.$$

(b) (i) We can have

$$|\boldsymbol{A_1}| = 1$$

$$|\boldsymbol{A_2}| = \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 3$$

$$|\boldsymbol{A_3}| = 3 \cdot |\boldsymbol{A_2}| + |\boldsymbol{A_1}| = 3 \cdot 3 + 1 = 10$$

$$|\boldsymbol{A_4}| = 4 \cdot |\boldsymbol{A_3}| + |\boldsymbol{A_2}| = 4 \cdot 10 + 3 = 43$$

$$|\boldsymbol{A_5}| = 5 \cdot |\boldsymbol{A_4}| + |\boldsymbol{A_3}| = 5 \cdot 43 + 10 = 225$$

$$|\boldsymbol{A_6}| = 6 \cdot |\boldsymbol{A_5}| + |\boldsymbol{A_4}| = 6 \cdot 225 + 43 = 1393.$$

(ii) We can have

$$
\begin{aligned}
|\boldsymbol{A_1}| &= 5 \\
|\boldsymbol{A_2}| &= \begin{vmatrix} 5 & -1 \\ 1 & 4 \end{vmatrix} = 21 \\
|\boldsymbol{A_3}| &= 3 \cdot |\boldsymbol{A_2}| + |\boldsymbol{A_1}| = 3 \cdot 21 + 5 = 68 \\
|\boldsymbol{A_4}| &= 2 \cdot |\boldsymbol{A_3}| + |\boldsymbol{A_2}| = 2 \cdot 68 + 21 = 157 \\
|\boldsymbol{A_5}| &= 1 \cdot |\boldsymbol{A_4}| + |\boldsymbol{A_3}| = 1 \cdot 157 + 68 = 225 \\
|\boldsymbol{A_6}| &= 0 \cdot |\boldsymbol{A_5}| + |\boldsymbol{A_4}| = 0 \cdot 225 + 157 = 157.
\end{aligned}
$$