EE 2030 Linear Algebra — Chi-chao Chao
Spring 2011

Final Examination

7:00pm to 10:00pm, June 17, 2011

Problems for Solution:

1. (20%) True or false. (If the statement is true, prove it. Otherwise, find a counterexample or explain why it is false.)

   (a) (5%) If $\boldsymbol{Q}$ is an orthogonal matrix, then the determinant of $\boldsymbol{Q}$ is 1.

   (b) (5%) $\frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & -5 \end{bmatrix}$ is a projection matrix.

   (c) (5%) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

   (d) (5%) If $\boldsymbol{A}$ is any m by n matrix with $m > n$, then $\boldsymbol{A}\boldsymbol{A}^T$ cannot be positive definite. (*Hint:* Consider the rank of $\boldsymbol{A}$.)

2. (20%) Consider

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

   (a) (5%) Find the $\boldsymbol{LU}$ decomposition of $\boldsymbol{A}$, where $\boldsymbol{L}$ is lower triangular and $\boldsymbol{U}$ is upper triangular.

   (b) (5%) Find the $\boldsymbol{QR}$ decomposition of $\boldsymbol{A}$, where $\boldsymbol{Q}$ is orthogonal and $\boldsymbol{R}$ is upper triangular.

   (c) (5%) Find the $\boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T$ decomposition of $\boldsymbol{A}$, where $\boldsymbol{Q}$ is orthogonal and $\boldsymbol{\Lambda}$ is diagonal.

   (d) (5%) Find the Cholesky ($\boldsymbol{C}\boldsymbol{C}^T$) decomposition of $\boldsymbol{A}$, where $\boldsymbol{C}$ is lower triangular with positive diagonal entries.

3. (a) (10%) Suppose x_k is the fraction of Electrical Engineering students at National Tsing Hua University who prefer calculus to linear algebra at year k. The remaining fraction $y_k = 1 - x_k$ prefers linear algebra. At year $k+1$, 1/5 of those who prefer calculus change their mind (possibly after taking EE 2030). Also at year $k+1$, 1/10 of those who prefer linear algebra change their mind (possibly because of the final exam). Create the matrix $\boldsymbol{A}$ to give

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} x_k \\ y_k \end{bmatrix}.$$

   If initially $x_0 = 1$, find the limit of y_k as $k \to \infty$.

(b) (10%) Solve for $x(t)$ and $y(t)$ in these differential equations, starting from $x(0) = 1$, $y(0) = 0$:

$$\frac{dx}{dt} = 3x - 4y, \quad \frac{dy}{dt} = 2x - 3y.$$

4. (20%) Consider

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

(a) (5%) Is $\boldsymbol{A}$ diagonalizable? If yes, find an invertible matrix $\boldsymbol{S}$ and a diagonal matrix $\boldsymbol{\Lambda}$ such that $\boldsymbol{A} = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^{-1}$. Otherwise, explain why it is not.

(b) (5%) Find the Jordan form $\boldsymbol{J}$ for $\boldsymbol{A}$.

(c) (5%) Find the singular value decomposition of $\boldsymbol{A}$.

(d) (5%) Find orthonormal bases for the nullspace and the left nullspace of $\boldsymbol{A}$.

5. (20%) Let P_2 be the space of all polynomials of degree at most 2, i.e., $P_2 = \{a_0 + a_1x + a_2x^2 : a_0, a_1, a_2 \in \mathcal{R}\}$. Consider the linear operator L on P_2 defined by

$$L(p(x)) = xp'(x) + p''(x)$$

where $p'(x)$ is the derivative of $p(x)$ and $p''(x)$ is the second derivative of $p(x)$.

(a) (5%) Find the matrix $\boldsymbol{A}$ representing L with respect to the basis $\{1, x, x^2\}$.

(b) (5%) Find the matrix $\boldsymbol{B}$ representing L with respect to the basis $\{1, x, 1 + x^2\}$.

(c) (5%) Find the matrix $\boldsymbol{M}$ such that $\boldsymbol{B} = \boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}$.

(d) (5%) If $p(x) = b_0 + b_1x + b_2(1 + x^2)$, calculate $L^n(p(x))$, where $L^1(p(x)) = L(p(x))$ and $L^n(p(x)) = L(L^{n-1}(p(x)))$ for $n > 1$.