Solution to Final Examination

1. (a) False. For example,

$$\boldsymbol{Q} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

is an orthogonal matrix since $\boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I}$, but we have $\det(\boldsymbol{Q}) = -1$.

(b) False. Let

$$\boldsymbol{A} = \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & -5 \end{bmatrix}.$$

We have

$$\boldsymbol{A}^2 = \frac{1}{36}\begin{bmatrix} 30 & 12 & 4 \\ 12 & 12 & -8 \\ 4 & -8 & 30 \end{bmatrix} \neq \boldsymbol{A}.$$

Therefore, $\boldsymbol{A}$ is not a projection matrix.

(c) True. Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and

$$\boldsymbol{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

We can find the eigenvalues of $\boldsymbol{A}$ with

$$\begin{aligned} \det(\boldsymbol{A} - \lambda \boldsymbol{I}) &= \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} \\ &= -\lambda^2(\lambda - 3) = 0. \end{aligned}$$

Therefore, the eigenvalues of $\boldsymbol{A}$ are $0, 0, 3$. Since $\boldsymbol{A}$ is symmetric, it is diagonalizable and is similar to

$$\boldsymbol{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

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(d) True. Let r be the rank of $\boldsymbol{A}$, and then $r \le n < m$. Since $\boldsymbol{A}\boldsymbol{A}^T$ is m by m and $\text{rank}(\boldsymbol{A}\boldsymbol{A}^T) = \text{rank}(\boldsymbol{A}) = r < m$, $\boldsymbol{A}\boldsymbol{A}^T$ is singular. Therefore, $\det(\boldsymbol{A}\boldsymbol{A}^T) = 0$, which yields that $\boldsymbol{A}\boldsymbol{A}^T$ cannot be positive definite.

2. (a) By Gaussian elimination, we can obtain

$$\begin{bmatrix} 1 & 0 \\ -1/2 & 1 \end{bmatrix} \boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 0 & 3/2 \end{bmatrix}.$$

Therefore,

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3/2 \end{bmatrix} = \boldsymbol{LU}.$$

(b) Since the columns of $\boldsymbol{A}$ are independent, let $\boldsymbol{a}_1 = (2,1)^T$ and $\boldsymbol{a}_2 = (1,2)^T$. By the Gram-Schmidt process, we can have

$$\boldsymbol{A}_1 = \boldsymbol{a}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \implies \boldsymbol{q}_1 = \frac{\boldsymbol{A}_1}{\|\boldsymbol{A}_1\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\boldsymbol{A}_2 = \boldsymbol{a}_2 - (\boldsymbol{a}_2{}^T \boldsymbol{q}_1)\boldsymbol{q}_1 = \begin{bmatrix} -3/5 \\ 6/5 \end{bmatrix} \implies \boldsymbol{q}_2 = \frac{\boldsymbol{A}_2}{\|\boldsymbol{A}_2\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Therefore,

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1{}^T \boldsymbol{a}_1 & \boldsymbol{q}_1{}^T \boldsymbol{a}_2 \\ 0 & \boldsymbol{q}_2{}^T \boldsymbol{a}_2 \end{bmatrix} = \boldsymbol{QR}$$

which gives

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 5/\sqrt{5} & 4/\sqrt{5} \\ 0 & 3/\sqrt{5} \end{bmatrix}.$$

(c) We can have

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda)$$

$$\implies \begin{cases} \lambda_1 = 3 \iff \boldsymbol{v}_1 = \frac{1}{\sqrt{2}}(1,1)^T \\ \lambda_2 = 1 \iff \boldsymbol{v}_2 = \frac{1}{\sqrt{2}}(1,-1)^T. \end{cases}$$

Therefore,

$$\boldsymbol{A} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T = \boldsymbol{Q\Lambda Q}^T.$$

(d) From (a), we can obtain

$$\begin{aligned} \boldsymbol{A} &= \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3/2 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{6}/2 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{6}/2 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}^T \\ &= \begin{bmatrix} \sqrt{2} & 0 \\ \sqrt{2}/2 & \sqrt{6}/2 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ \sqrt{2}/2 & \sqrt{6}/2 \end{bmatrix}^T = \boldsymbol{CC}^T. \end{aligned}$$

3. (a) Let $\boldsymbol{u}_k = \begin{bmatrix} x_k \\ y_k \end{bmatrix}$. The relation between $\boldsymbol{u}_{k+1} = \begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix}$ and $\boldsymbol{u}_k = \begin{bmatrix} x_k \\ y_k \end{bmatrix}$ is given by

$$\boldsymbol{u}_{k+1} = \begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} 4/5 & 1/10 \\ 1/5 & 9/10 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} = \boldsymbol{A}\boldsymbol{u}_k.$$

To find $\boldsymbol{A}^k$, we first find the eigenvalues and corresponding eigenvectors of $\boldsymbol{A}$.

$$\begin{aligned} \det(\boldsymbol{A} - \lambda \boldsymbol{I}_2) &= \begin{vmatrix} (4/5) - \lambda & 1/10 \\ 1/5 & (9/10) - \lambda \end{vmatrix} \\ &= \lambda^2 - \frac{17}{10}\lambda + \frac{7}{10} \\ &= (\lambda - 1)(\lambda - (7/10)) = 0 \end{aligned}$$

$$\Longrightarrow \begin{cases} \lambda_1 = 1 & \Longleftrightarrow \quad \boldsymbol{v}_1 = (1, 2)^T \\ \lambda_2 = 7/10 & \Longleftrightarrow \quad \boldsymbol{v}_2 = (1, -1)^T. \end{cases}$$

Therefore, we have

$$\boldsymbol{A} = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 7/10 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}^{-1}.$$

We can write $\boldsymbol{u}_0$ as a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ as follows:

$$\boldsymbol{u}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{aligned} &\Longrightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} \\ &\Longrightarrow \boldsymbol{u}_0 = \frac{1}{3}\boldsymbol{v}_1 + \frac{2}{3}\boldsymbol{v}_2. \end{aligned}$$

Then we can obtain

$$\begin{aligned} \boldsymbol{u}_k &= \boldsymbol{A}^k \boldsymbol{u}_0 \\ &= \frac{1}{3}\boldsymbol{A}^k \boldsymbol{v}_1 + \frac{2}{3}\boldsymbol{A}^k \boldsymbol{v}_2 \\ &= \frac{1}{3}\boldsymbol{v}_1 + \frac{2}{3}\left(\frac{7}{10}\right)^k \boldsymbol{v}_2 \\ &= \begin{bmatrix} (1/3) + (2/3)(7/10)^k \\ (2/3) - (2/3)(7/10)^k \end{bmatrix}. \end{aligned}$$

Therefore,

$$\lim_{k\to\infty} y_k = \lim_{k\to\infty} \left(\frac{2}{3} - \frac{2}{3}\left(\frac{7}{10}\right)^k \right) = \frac{2}{3}$$

i.e., after a long time, 2/3 of the NTHUEE students prefer linear algebra to calculus.

(b) Let $\boldsymbol{u} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. We then have

$$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix} \boldsymbol{u} = \boldsymbol{A}\boldsymbol{u}.$$

To find the eigenvalues and corresponding eigenvectors of $\boldsymbol{A}$, we calculate

$$\begin{aligned} \det(\boldsymbol{A} - \lambda \boldsymbol{I}_2) &= \begin{vmatrix} 3-\lambda & -4 \\ 2 & -3-\lambda \end{vmatrix} \\ &= \lambda^2 - 1 \\ &= (\lambda - 1)(\lambda + 1) = 0 \end{aligned}$$

$$\Longrightarrow \begin{cases} \lambda_1 = 1 & \Longleftrightarrow \quad \boldsymbol{v}_1 = (2,1)^T \\ \lambda_2 = -1 & \Longleftrightarrow \quad \boldsymbol{v}_2 = (1,1)^T. \end{cases}$$

Hence, we can obtain

$$\boldsymbol{u} = c_1 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2c_1 e^t + c_2 e^{-t} \\ c_1 e^t + c_2 e^{-t} \end{bmatrix}.$$

At $t = 0$, we have

$$\begin{cases} x(0) = 1 = 2c_1 + c_2 \\ y(0) = 0 = c_1 + c_2 \end{cases} \Longrightarrow \begin{cases} c_1 = 1 \\ c_2 = -1. \end{cases}$$

Therefore,

$$x(t) = 2e^t - e^{-t}$$

and

$$y(t) = e^t - e^{-t}.$$

4. (a) A matrix $\boldsymbol{A}$ is diagonalizable if and only if each of its eigenvalues has the same algebraic multiplicity (AM) and geometric multiplicity (GM). Hence we should find the eigenvalues and corresponding eigenvectors of $\boldsymbol{A}$. Since

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{bmatrix} -\lambda & 0 & 0 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix} = -\lambda^3 = 0$$

we have $\lambda = 0, 0, 0$. For $\lambda = 0$, the AM of λ is 3. Besides,

$$\boldsymbol{A} - \lambda \boldsymbol{I} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and the corresponding eigenvector is

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This gives the GM of λ is 1, which is smaller than the AM of λ. As a result, $\boldsymbol{A}$ is not diagonalizable.

(b) Since the AM and GM of the eigenvalue 0 are 3 and 1, respectively, the Jordan form $\boldsymbol{J}$ for $\boldsymbol{A}$ is

$$\boldsymbol{J} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) To obtain the singular value decomposition of $\boldsymbol{A}$, we first find the eigenvalues and corresponding orthonormal eigenvectors of $\boldsymbol{A}^T\boldsymbol{A}$. After some calculations, we can obtain

$$\boldsymbol{A}^T\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{cases} \lambda_1 = 1,\ \lambda_2 = 1 & \longleftrightarrow \quad \boldsymbol{v}_1 = (1,0,0)^T,\ \boldsymbol{v}_2 = (0,1,0)^T \\ \lambda_3 = 0 & \longleftrightarrow \quad \boldsymbol{v}_3 = (0,0,1)^T. \end{cases}$$

For $\lambda_1 = \lambda_2 = 1$, we have the singular values $\sigma_1 = \sqrt{\lambda_1} = 1$, $\sigma_2 = \sqrt{\lambda_2} = 1$, and

$$\boldsymbol{u}_1 = \frac{\boldsymbol{A}\boldsymbol{v}_1}{\sigma_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \boldsymbol{u}_2 = \frac{\boldsymbol{A}\boldsymbol{v}_2}{\sigma_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

By the Gram-Schmidt process, we can obtain

$$\boldsymbol{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

As a result, we have the singular value decomposition of $\boldsymbol{A}$ as

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T \\ &= \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix}^T \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

(d) According to what was taught in class, we have $\boldsymbol{v}_3 = (0,0,1)^T$ and $\boldsymbol{u}_3 = (1,0,0)^T$ form an orthonormal basis for $\mathcal{N}(\boldsymbol{A})$ and $\mathcal{N}(\boldsymbol{A}^T)$, respectively.

5. (a) Let $\beta = \{1, x, x^2\}$. Since

$$\begin{aligned} L(1) &= 0 &&= 0\cdot 1 + 0\cdot x + 0\cdot x^2 \\ L(x) &= x &&= 0\cdot 1 + 1\cdot x + 0\cdot x^2 \\ L(x^2) &= 2x^2 + 2 &&= 2\cdot 1 + 0\cdot x + 2\cdot x^2 \end{aligned}$$

we have

$$\boldsymbol{A} = [L]_\beta = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(b) Let $\gamma = \{1, x, 1+x^2\}$. Since

$$\begin{array}{rcl} L(1) &=& 0 &=& 0\cdot 1 + 0\cdot x + 0\cdot(1+x^2) \\ L(x) &=& x &=& 0\cdot 1 + 1\cdot x + 0\cdot(1+x^2) \\ L(1+x^2) &=& 2x^2+2 &=& 0\cdot 1 + 0\cdot x + 2\cdot(1+x^2) \end{array}$$

we have

$$\boldsymbol{B} = [L]_\gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(c) Let I be the identity transformation, i.e., $I(p(x)) = p(x)$. According to what was taught in class, we have $\boldsymbol{M} = [I]_\gamma^\beta$. Since

$$\begin{array}{rcl} I(1) &=& 1 &=& 1\cdot 1 + 0\cdot x + 0\cdot x^2 \\ I(x) &=& x &=& 0\cdot 1 + 1\cdot x + 0\cdot x^2 \\ I(1+x^2) &=& 1+x^2 &=& 1\cdot 1 + 0\cdot x + 1\cdot x^2 \end{array}$$

we can obtain

$$\boldsymbol{M} = [I]_\gamma^\beta = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(d) Using γ as the basis for P_2, we can have

$$p(x) = b_0\cdot 1 + b_1\cdot x + b_2\cdot(1+x^2) \Longleftrightarrow \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}$$

and

$$L(p(x)) \Longleftrightarrow [L]_\gamma \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}.$$

To find $L^n(p(x))$, we should diagonalize $[L]_\gamma$. Since $[L]_\gamma$ is already a diagonal matrix, we can obtain

$$([L]_\gamma)^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{bmatrix}.$$

Therefore,

$$([L]_\gamma)^n \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ b_1 \\ 2^n b_2 \end{bmatrix}$$

$$\begin{aligned} \Longrightarrow L^n(p(x)) &= 0 \cdot 1 + b_1 \cdot x + 2^n b_2 \cdot (1 + x^2) \\ &= 2^n b_2 + b_1 x + 2^n b_2 x^2. \end{aligned}$$