EE 2030 Linear Algebra Chi-chao Chao
Spring 2011

Midterm Examination No. 1
7:00pm to 10:00pm, April 1, 2011

Problems for Solution:

1. (10%) For each of the matrices

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 4 & 6 \\ 0 & 3 & 8 & 5 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 9 \end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/3 & 1/3 & 1 & 0 \\ 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}$$

determine whether the matrix is invertible. If so, find its inverse.

2. (10%) Find the $\boldsymbol{PA} = \boldsymbol{LU}$ factorization for

$$\boldsymbol{A} = \begin{bmatrix} 0 & 2 & 2 & 4 \\ 0 & 2 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 6 & 7 & 5 \end{bmatrix}$$

where $\boldsymbol{P}$ is a permutation matrix, $\boldsymbol{L}$ is a lower triangular matrix with unit diagonal, and $\boldsymbol{U}$ is an upper triangular matrix.

3. (10%) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric matrices, which of the following matrices are certainly symmetric? (You need to justify your answers.)

   (a) $\boldsymbol{A}^2 - \boldsymbol{B}^2$.

   (b) $(\boldsymbol{A} + \boldsymbol{B})(\boldsymbol{A} - \boldsymbol{B})$.

4. (15%) Is each of the following subsets of $\mathcal{R}^3$ actually a subspace? If yes, prove it. Otherwise, find a counterexample.

   (a) All vectors (b_1, b_2, b_3) such that $2b_1 - 2b_2 + b_3 = 0$.

   (b) All vectors (b_1, b_2, b_3) such that $2b_1 - 2b_2 + b_3 = 1$.

   (c) All vectors (b_1, b_2, b_3) such that $b_1 = b_2$ or $b_1 = 2b_3$.

5. (10%) Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 5 \\ 1 & 3 & 5 & 9 \end{bmatrix}.$$

(a) Find a matrix $\boldsymbol{B}$ such that the column space $\mathcal{C}(\boldsymbol{A})$ of $\boldsymbol{A}$ equals the nullspace $\mathcal{N}(\boldsymbol{B})$ of $\boldsymbol{B}$. (*Hint*: If $\boldsymbol{b} = (b_1, b_2, b_3)^T$ is in $\mathcal{C}(\boldsymbol{A})$, then what system of linear equations should $b_1,\ b_2,\ b_3$ satisfy?)

(b) Which of the following vectors belong to the column space $\mathcal{C}(\boldsymbol{A})$:

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}?$$

6. (10%) Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 4 & k \end{bmatrix}.$$

For all values of k, find the complete solution to $\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}$. (You might have to consider several cases.)

7. (10%) Suppose $\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \boldsymbol{v}_3$ are independent vectors. Let

$$\begin{aligned} \boldsymbol{w}_1 &= a_{11}\boldsymbol{v}_1 + a_{12}\boldsymbol{v}_2 + a_{13}\boldsymbol{v}_3 \\ \boldsymbol{w}_2 &= a_{21}\boldsymbol{v}_1 + a_{22}\boldsymbol{v}_2 + a_{23}\boldsymbol{v}_3 \\ \boldsymbol{w}_3 &= a_{31}\boldsymbol{v}_1 + a_{32}\boldsymbol{v}_2 + a_{33}\boldsymbol{v}_3. \end{aligned}$$

Under what conditions on a_{ij}, for $1 \le i, j \le 3$, will $\boldsymbol{w}_1,\ \boldsymbol{w}_2,\ \boldsymbol{w}_3$ also be independent? (You need to justify your answer.)

8. (10%) Let S and T are subspaces of $\mathcal{R}^4$ given by

$$S = \{(a, b, c, d) : a + c + d = 0\}$$
$$T = \{(a, b, c, d) : a + b = 0,\ c = 2d\}$$

respectively.

(a) Find a basis for S.

(b) What is the dimension of the intersection $S \cap T$?

9. (15%) This problem is about a 3 by 2 matrix $\boldsymbol{A}$ for which $\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has **no solution** and $\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has **exactly one solution**.

(a) What is the rank of $\boldsymbol{A}$?

(b) Find all solutions to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$.

(c) Write down an example of $\boldsymbol{A}$.