Solution to Midterm Examination No. 1

1. (a) Using the Gauss-Jordan method, we can have

$$\left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{I} \end{array}\right] = \left[\begin{array}{cccc|cccc} 2 & 1 & 4 & 6 & 1 & 0 & 0 & 0 \\ 0 & 3 & 8 & 5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 9 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{cccc|cccc} 2 & 1 & 4 & 6 & 1 & 0 & 0 & 0 \\ 0 & 3 & 8 & 5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -9/7 & 1 \end{array}\right].$$

Since we cannot obtain four nonzero pivots, $\boldsymbol{A}$ is not invertible.

(b) Using the Gauss-Jordan method, we can have

$$\left[\begin{array}{c|c} \boldsymbol{B} & \boldsymbol{I} \end{array}\right] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1/3 & 1/3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1/2 & 1/2 & 1/2 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/4 & 1 & 0 & 0 \\ 0 & 1/3 & 1 & 0 & -1/3 & 0 & 1 & 0 \\ 0 & 1/2 & 1/2 & 1 & -1/2 & 0 & 0 & 1 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1/4 & -1/3 & 1 & 0 \\ 0 & 0 & 1/2 & 1 & -3/8 & -1/2 & 0 & 1 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1/4 & -1/3 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1/4 & -1/3 & -1/2 & 1 \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{I} & \boldsymbol{B}^{-1} \end{array}\right].$$

Hence, $\boldsymbol{B}$ is invertible and the inverse is given by

$$\boldsymbol{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/4 & 1 & 0 & 0 \\ -1/4 & -1/3 & 1 & 0 \\ -1/4 & -1/3 & -1/2 & 1 \end{bmatrix}.$$

2. First do row exchanges as

$$\boldsymbol{A} = \begin{bmatrix} 0 & 2 & 2 & 4 \\ 0 & 2 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 6 & 7 & 5 \end{bmatrix} \overset{\boldsymbol{P}}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 2 & 2 & 4 \\ 2 & 6 & 7 & 5 \\ 0 & 2 & 2 & 2 \end{bmatrix} = \boldsymbol{PA}$$

and then perform eliminations as

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 2 & 2 & 4 \\ 2 & 6 & 7 & 5 \\ 0 & 2 & 2 & 2 \end{bmatrix} \overset{\boldsymbol{E}_{31}}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 2 & 2 & 4 \\ 0 & 2 & 3 & 3 \\ 0 & 2 & 2 & 2 \end{bmatrix} \overset{\boldsymbol{E}_{32}}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 2 & 2 & 2 \end{bmatrix} \overset{\boldsymbol{E}_{42}}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix} = \boldsymbol{U}.$$

Then we have

$$\boldsymbol{E}_{42}\boldsymbol{E}_{32}\boldsymbol{E}_{31}(\boldsymbol{PA}) = \boldsymbol{U}$$

where

$$\boldsymbol{P} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{E}_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{E}_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } \boldsymbol{E}_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

We can obtain

$$\boldsymbol{L} = \boldsymbol{E}_{31}^{-1}\boldsymbol{E}_{32}^{-1}\boldsymbol{E}_{42}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

The factorization $\boldsymbol{PA} = \boldsymbol{LU}$ is hence given by

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 4 \\ 0 & 2 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 6 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

3. (a) Since $\boldsymbol{A}^T = \boldsymbol{A}$ and $\boldsymbol{B}^T = \boldsymbol{B}$, we have

$$\begin{aligned} (\boldsymbol{A}^2 - \boldsymbol{B}^2)^T &= (\boldsymbol{A}^2)^T - (\boldsymbol{B}^2)^T \\ &= (\boldsymbol{AA})^T - (\boldsymbol{BB})^T \\ &= \boldsymbol{A}^T\boldsymbol{A}^T - \boldsymbol{B}^T\boldsymbol{B}^T \\ &= (\boldsymbol{A}^T)^2 - (\boldsymbol{B}^T)^2 \\ &= \boldsymbol{A}^2 - \boldsymbol{B}^2. \end{aligned}$$

Therefore, $\boldsymbol{A}^2 - \boldsymbol{B}^2$ is certainly symmetric.

(b) Consider

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \text{ and } \boldsymbol{B} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Then

$$\begin{aligned} (\boldsymbol{A}+\boldsymbol{B})(\boldsymbol{A}-\boldsymbol{B}) &= \left(\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right)\left(\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right) \\ &= \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} \end{aligned}$$

which is not symmetric. Therefore, $(\boldsymbol{A}+\boldsymbol{B})(\boldsymbol{A}-\boldsymbol{B})$ is not certainly symmetric.

4. (a) Yes, this subset is a subspace of $\mathcal{R}^3$. Let $\mathcal{B}_1 = \{(b_1, b_2, b_3) : 2b_1 - 2b_2 + b_3 = 0\}$. Take two vectors $\boldsymbol{u} = (u_1, u_2, u_3) \in \mathcal{B}_1$ and $\boldsymbol{v} = (v_1, v_2, v_3) \in \mathcal{B}_1$, where $2u_1 - 2u_2 + u_3 = 0$ and $2v_1 - 2v_2 + v_3 = 0$. Then we check the following two conditions:

- Consider $\boldsymbol{u}+\boldsymbol{v} = (u_1+v_1, u_2+v_2, u_3+v_3)$. Since $2(u_1+v_1) - 2(u_2+v_2) + (u_3+v_3) = (2u_1 - 2u_2 + u_3) + (2v_1 - 2v_2 + v_3) = 0$, $\boldsymbol{u}+\boldsymbol{v} \in \mathcal{B}_1$.
- For any $c \in \mathcal{R}$, consider $c\boldsymbol{u} = (cu_1, cu_2, cu_3)$. Since $2cu_1 - 2cu_2 + cu_3 = c(2u_1 - 2u_2 + u_3) = 0$, $c\boldsymbol{u} \in \mathcal{B}_1$.

Therefore, $\mathcal{B}_1$ is a subspace of $\mathcal{R}^3$.

(b) No, this subset is not a subspace of $\mathcal{R}^3$. Let $\mathcal{B}_2 = \{(b_1, b_2, b_3) : 2b_1 - 2b_2 + b_3 = 1\}$. Consider $(0,0,1) \in \mathcal{B}_2$ and $(1/2, 0, 0) \in \mathcal{B}_2$. Since $(0,0,1) + (1/2,0,0) = (1/2, 0, 1) \notin \mathcal{B}_2$, $\mathcal{B}_2$ is not a subspace of $\mathcal{R}^3$.

(c) No, this subset is not a subspace of $\mathcal{R}^3$. Let $\mathcal{B}_3 = \{(b_1, b_2, b_3) : b_1 = b_2 \text{ or } b_1 = 2b_3\}$. Consider $(2,2,0) \in \mathcal{B}_3$ and $(2,0,1) \in \mathcal{B}_3$. Since $(2,2,0) + (2,0,1) = (4,2,1) \notin \mathcal{B}_3$, $\mathcal{B}_3$ is not a subspace of $\mathcal{R}^3$.

5. (a) Since $\boldsymbol{b} = (b_1, b_2, b_3)^T$ is in $\mathcal{C}(\boldsymbol{A})$, $\boldsymbol{Ax} = \boldsymbol{b}$ must be solvable. We can have

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & b_1 \\ 1 & 2 & 3 & 5 & b_2 \\ 1 & 3 & 5 & 9 & b_3 \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & b_1 \\ 0 & 1 & 2 & 4 & b_2 - b_1 \\ 0 & 2 & 4 & 8 & b_3 - b_1 \end{array}\right]$$
$$\Longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & b_1 \\ 0 & 1 & 2 & 4 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_1 - 2b_2 + b_3 \end{array}\right].$$

For $\boldsymbol{Ax} = \boldsymbol{b}$ to be solvable, we should have

$$b_1 - 2b_2 + b_3 = 0$$

which is

$$\begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = 0.$$

Hence, $\boldsymbol{B}$ can be chosen as

$$\begin{bmatrix} 1 & -2 & 1 \end{bmatrix}.$$

(b) For $\boldsymbol{b} = (b_1, b_2, b_3)^T$ to be in $\mathcal{C}(\boldsymbol{A})$, from (a) we must have

$$b_1 - 2b_2 + b_3 = 0.$$

Therefore,

$$\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \in \mathcal{C}(\boldsymbol{A})$$

and the rest are not in $\mathcal{C}(\boldsymbol{A})$.

6. To find the complete solution, we reduce the matrix to the RRE form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 3 \\ 3 & 4 & k & 7 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & k-3 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & k-5 & 0 \end{array}\right].$$

- If $k = 5$, then we have

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, the pivot variables are x_1 and x_2, and the free variable is x_3. Setting $x_3 = 0$, we can obtain $x_1 = 1$ and $x_2 = 1$. Therefore, a particular solution can be given by

$$\boldsymbol{x}_p = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

To find the special soultion $\boldsymbol{x}_n$, we let

$$\begin{cases} x_1 \quad - \quad x_3 = 0 \\ \quad x_2 + 2x_3 = 0. \end{cases}$$

Setting $x_3 = 1$, we have $x_1 = 1$, $x_2 = -2$. Therefore, a special solution $\boldsymbol{x}_n$ is

$$\boldsymbol{x}_n = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

As a result, the complete solution to this problem is

$$\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

- If $k \neq 5$, x_3 must be equal to 0. Then we can obtain $x_1 = 1$ and $x_2 = 1$. The solution to this problem is

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

7. Consider the following equation:

$$\begin{aligned} & x_1\boldsymbol{w}_1 + x_2\boldsymbol{w}_2 + x_3\boldsymbol{w}_3 = 0 \\ \Longrightarrow\ & x_1(a_{11}\boldsymbol{v}_1 + a_{12}\boldsymbol{v}_2 + a_{13}\boldsymbol{v}_3) + x_2(a_{21}\boldsymbol{v}_1 + a_{22}\boldsymbol{v}_2 + a_{23}\boldsymbol{v}_3) \\ & \qquad + x_3(a_{31}\boldsymbol{v}_1 + a_{32}\boldsymbol{v}_2 + a_{33}\boldsymbol{v}_3) = 0 \\ \Longrightarrow\ & (x_1a_{11} + x_2a_{21} + x_3a_{31})\boldsymbol{v}_1 + (x_1a_{12} + x_2a_{22} + x_3a_{32})\boldsymbol{v}_2 \\ & \qquad + (x_1a_{13} + x_2a_{23} + x_3a_{33})\boldsymbol{v}_3 = 0. \end{aligned}$$

Since $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ are linearly independent, we know the only solution to the above equation is

$$\begin{cases} x_1a_{11} + x_2a_{21} + x_3a_{31} = 0 \\ x_1a_{12} + x_2a_{22} + x_3a_{32} = 0 \\ x_1a_{13} + x_2a_{23} + x_3a_{33} = 0 \end{cases}$$

$$\Longrightarrow \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \boldsymbol{Ax} = \mathbf{0}$$

where $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$ and $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. If $\boldsymbol{w}_1$, $\boldsymbol{w}_2$, $\boldsymbol{w}_3$ are independent, then $x_1\boldsymbol{w}_1 + x_2\boldsymbol{w}_2 + x_3\boldsymbol{w}_3 = 0$ only if $x_1 = x_2 = x_3 = 0$, and hence $\boldsymbol{Ax} = \mathbf{0}$ has the only one solution $\mathbf{0}$. This implies $\boldsymbol{A}$ must be invertible, i.e., $\boldsymbol{A}$ has full rank.

8. (a) $\mathcal{S} = \{(a,b,c,d) : a + c + d = 0, a,b,c,d \in \mathcal{R}\}$. We can obtain

$$\begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0.$$

Hence, $\mathcal{S}$ is the nullspace of $\begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}$. Since b, c, d are free variables, special solutions are given by

$$(a,b,c,d) = (0,1,0,0)$$
$$(a,b,c,d) = (-1,0,1,0)$$
$$(a,b,c,d) = (-1,0,0,1).$$

The special solutions obtained are independent and span the nullspace. They form a basis for $\mathcal{S}$. Therefore, we have a basis:

$$(0,1,0,0), (-1,0,1,0), (-1,0,0,1).$$

(b) Consider $\mathcal{S} \cap \mathcal{T}$, and we can have

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

$$\Longrightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0.$$

Since $\mathcal{S} \cap \mathcal{T}$ is the nullspace of $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$, the dimension of $\mathcal{S} \cap \mathcal{T}$ is equal to $n - \text{rank}\left(\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}\right) = 4 - 3 = 1.$

9. Since $\boldsymbol{Ax} = \boldsymbol{b}$ has zero or one solution, we know that $\boldsymbol{A}$ has full column rank.

(a) Since $\boldsymbol{A}$ is a 3 by 2 matrix with full column rank, the rank of $\boldsymbol{A}$ is 2.

(b) Since $\boldsymbol{A}$ has full column rank, the dimension of $\mathcal{N}(\boldsymbol{A})$ is 0. Therefore, $\boldsymbol{Ax} = \boldsymbol{0}$ has only one solution $\boldsymbol{x} = \boldsymbol{0}$.

(c) Consider

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}.$$

Then $\boldsymbol{A}$ has full column rank. It can be easily checked that $\boldsymbol{Ax} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has no solution and $\boldsymbol{Ax} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has exactly one solution $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.