EE 2030 Linear Algebra Chi-chao Chao
Spring 2011

Midterm Examination No. 2
7:00pm to 10:00pm, May 6, 2011

Problems for Solution:

1. (15%) Write down a matrix with the required property or explain why no such matrix exists.

   (a) (5%) Column space contains $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, row space contains $(1, 1)$, $(1, 2)$.

   (b) (5%) Column space has basis $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, nullspace has basis $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

   (c) (5%) Column space = $\mathcal{R}^4$, row space = $\mathcal{R}^3$. ($\mathcal{R}$ is the set of real numbers.)

2. (20%) Consider

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 3 & 5 \\ -1 & -3 & 1 & 0 \end{bmatrix}.$$

   (a) (5%) Find a basis for the row space of $\boldsymbol{A}$.

   (b) (5%) Find a basis for the orthogonal complement of the column space of $\boldsymbol{A}$.

   (c) (5%) Find the projection matrix $\boldsymbol{P}_c$ onto the column space of $\boldsymbol{A}$.

   (d) (5%) Given $\boldsymbol{x} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$, split it into $\boldsymbol{x} = \boldsymbol{x}_c + \boldsymbol{x}_{ln}$, where $\boldsymbol{x}_c$ is in the column space of $\boldsymbol{A}$ and $\boldsymbol{x}_{ln}$ is in the left nullspace of $\boldsymbol{A}$.

3. (15%) We have four data points with measurements $b = 2, 0, -3, -5$ at times $t = -1, 0, 1, 2$.

   (a) (5%) Suppose we want to fit the four data points with a horizontal line: $b = C_1$. Find the best least squares horizontal line fit.

   (b) (5%) Suppose we want to fit the four data points with a straight line: $b = C_2 + D_2 t$. Find the best least squares straight line fit.

   (c) (5%) Suppose we want to fit the four data points with a parabola: $b = C_3 + D_3 t + E_3 t^2$. Find the best least squares parabola fit.

4. (15%) Consider the vector space $C[-2,2]$, the space of all real-valued continuous functions on $[-2,2]$, with inner product defined by

$$\langle f, g\rangle = \int_{-2}^{2} f(x)g(x)\,dx.$$

   (a) (10%) Find an orthonormal basis for the subspace spanned by 1, x, and x^2.

   (b) (5%) Express x^2+2x as a linear combination of those orthonormal basis functions found in (a).

5. (15%) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be n by n real matrices. Is each of the following statements true or false? If it is true, prove it. Otherwise, find a counterexample.

   (a) (5%) If $\boldsymbol{A}$ is not invertible, then $\boldsymbol{AB}$ is not invertible.

   (b) (5%) The determinant of $\boldsymbol{A}-\boldsymbol{B}$ equals $\det \boldsymbol{A} - \det \boldsymbol{B}$.

   (c) (5%) A skew-symmetric matrix $\boldsymbol{A}$ has $\det \boldsymbol{A} = 0$ if n is odd. (Note that a skew-symmetric matrix satisfies $\boldsymbol{A}^T = -\boldsymbol{A}$.)

6. (10%) Let S_n be the determinant of the 1, 3, 1 tridiagonal matrix of order n:

$$S_1 = \begin{vmatrix} 3 \end{vmatrix}, \quad S_2 = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}, \quad S_3 = \begin{vmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix}, \quad S_4 = \begin{vmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{vmatrix}, \quad \ldots$$

   (a) (5%) Show that $S_n = aS_{n-1} + bS_{n-2}$, for $n \geq 3$. Find the constants a and b.

   (b) (5%) Find S_1, S_2, S_3, S_4, and S_5.

7. (10%) Consider the n by n matrix $\boldsymbol{A}_n$ that has zeros on its main diagonal and all other entries equal to 1, i.e.,

$$\boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{A}_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{A}_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad \ldots$$

   (a) (5%) Find the determinant of $\boldsymbol{A}_5$. (Here is a suggested approach: Start by adding all rows (except the last) to the last row, and then factoring out a constant.)

   (b) (5%) Find the $(1,1)$ entry of $\boldsymbol{A}_4^{-1}$.