Solution to Midterm Examination No. 2

1. (a) Consider the 3×2 matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $\mathcal{C}(\boldsymbol{A})$ contains $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\mathcal{C}(\boldsymbol{A}^T)$ contains $(1, 1)$, $(1, 2)$.

(b) Since the column space and nullspace both have three components, the desired matrix is 3 by 3, say $\boldsymbol{B}$. We can find $\dim(\mathcal{N}(\boldsymbol{B})) = 1 \neq 2 = 3 - 1 = 3 - \text{rank}(\boldsymbol{B})$, which is not possible. Therefore, no such matrix exists.

(c) Suppose the desired matrix exists. By the required property, column rank $= \dim(\mathcal{R}^4) = 4 \neq 3 = \dim(\mathcal{R}^3) =$ row rank, which is not possible. Therefore, no such matrix exists.

2. (a) Transform $\boldsymbol{A}$ into the RRE form:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 3 & 5 \\ -1 & -3 & 1 & 0 \end{bmatrix} \Longrightarrow \boldsymbol{R} = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, a basis for the row space of $\boldsymbol{A}$ can be given by

$$(1, 3, 0, 1), \ (0, 0, 1, 1).$$

(b) The orthogonal complement of the column space of $\boldsymbol{A}$ is the left nullspace of $\boldsymbol{A}$. We can have $\boldsymbol{R} = \boldsymbol{E}\boldsymbol{A}$ where

$$\boldsymbol{E} = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 0 \\ 5 & -2 & 1 \end{bmatrix}.$$

Since the last row of $\boldsymbol{R}$ is a zero row, a basis for the left nullspace can be given by the last row of $\boldsymbol{E}$:

$$(5, -2, 1).$$

(c) From $\boldsymbol{R}$, we know that $(1, 2, -1)^T$, $(1, 3, 1)^T$ form a basis for the column space of $\boldsymbol{A}$. Therefore, we can obtain

$$\begin{aligned} \boldsymbol{P}_c &= \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ -1 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ -1 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ -1 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ -1 & 1 \end{bmatrix}^T \\ &= \begin{bmatrix} 1/6 & 1/3 & -1/6 \\ 1/3 & 13/15 & 1/15 \\ -1/6 & 1/15 & 29/30 \end{bmatrix}. \end{aligned}$$

(d) We can project $\boldsymbol{x}$ onto the column space of $\boldsymbol{A}$ and obtain

$$\boldsymbol{x}_c \quad = \quad \boldsymbol{P}_c\boldsymbol{x} = \begin{bmatrix} 1/6 & 1/3 & -1/6 \\ 1/3 & 13/15 & 1/15 \\ -1/6 & 1/15 & 29/30 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Then we can have

$$\boldsymbol{x}_{ln} = \boldsymbol{x} - \boldsymbol{x}_c = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}.$$

3. (a) Let

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ \hat{\boldsymbol{x}}_1 = \begin{bmatrix} C_1 \end{bmatrix}, \text{ and } \boldsymbol{b}_1 = \begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix}.$$

We know the choice of $\hat{\boldsymbol{x}}_1$ which minimizes the squared error can be obtained by solving

$$\boldsymbol{A}_1^T\boldsymbol{A}_1\hat{\boldsymbol{x}}_1 = \boldsymbol{A}_1^T\boldsymbol{b}_1$$

which gives

$$4C_1 = -6.$$

Hence the best least squares horizontal line fit is given by $b = C_1 = -3/2$.

(b) Let

$$\boldsymbol{A}_2 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \ \hat{\boldsymbol{x}}_2 = \begin{bmatrix} C_2 \\ D_2 \end{bmatrix}, \text{ and } \boldsymbol{b}_2 = \begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix}.$$

The least squares solution $\hat{\boldsymbol{x}}_2$ can be obtained by solving

$$\boldsymbol{A}_2^T\boldsymbol{A}_2\hat{\boldsymbol{x}}_2 = \boldsymbol{A}_2^T\boldsymbol{b}_2$$

or equivalently,

$$\begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C_2 \\ D_2 \end{bmatrix} = \begin{bmatrix} -6 \\ -15 \end{bmatrix}.$$

Therefore, we have

$$\begin{bmatrix} C_2 \\ D_2 \end{bmatrix} = \begin{bmatrix} -3/10 \\ -12/5 \end{bmatrix}$$

and the best least squares straight line fit is given by $b = C_2 + D_2 t = -3/10 - (12/5)t$.

(c) Let

$$\boldsymbol{A}_3 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}, \ \hat{\boldsymbol{x}}_3 = \begin{bmatrix} C_3 \\ D_3 \\ E_3 \end{bmatrix}, \text{ and } \boldsymbol{b}_3 = \begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix}.$$

The least squares solution $\hat{\boldsymbol{x}}_3$ can again be obtained by solving

$$\boldsymbol{A}_3^T \boldsymbol{A}_3 \hat{\boldsymbol{x}}_3 = \boldsymbol{A}_3^T \boldsymbol{b}_3$$

which is given by

$$\begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix} \begin{bmatrix} C_3 \\ D_3 \\ E_3 \end{bmatrix} = \begin{bmatrix} -6 \\ -15 \\ -21 \end{bmatrix}.$$

Finally, we obtain

$$\begin{bmatrix} C_3 \\ D_3 \\ E_3 \end{bmatrix} = \begin{bmatrix} -3/10 \\ -12/5 \\ 0 \end{bmatrix}$$

and the best least squares parabola fit is given by $b = C_3 + D_3 t + E_3 t^2 = -3/10 - (12/5)t$. In this case, the best parabola fit is identical to the best straight line fit.

4. (a) Let $f_1(x) = 1$, $f_2(x) = x$, and $f_3(x) = x^2$. By the Gram-Schmidt process, we can obtain

$$\begin{aligned} & F_1(x) = f_1(x) = 1 \\ \Longrightarrow \quad & q_1(x) = \frac{F_1(x)}{\|F_1\|} = \frac{1}{\sqrt{\int_{-2}^{2} 1^2 dx}} = \frac{1}{2}. \end{aligned}$$

$$\begin{aligned} & F_2(x) = f_2(x) - \langle q_1, f_2 \rangle q_1(x) = x - \int_{-2}^{2} \frac{1}{2} x dx \cdot \frac{1}{2} = x \\ \Longrightarrow \quad & q_2(x) = \frac{F_2(x)}{\|F_2\|} = \frac{x}{\sqrt{\int_{-2}^{2} x^2 dx}} = \frac{\sqrt{3}}{4} x. \end{aligned}$$

$$\begin{aligned} F_3(x) &= f_3(x) - \langle q_1, f_3 \rangle q_1(x) - \langle q_2, f_3 \rangle q_2(x) \\ &= x^2 - \int_{-2}^{2} \frac{1}{2} x^2 dx \cdot \frac{1}{2} - \int_{-2}^{2} \frac{\sqrt{3}}{4} x^3 dx \cdot \frac{\sqrt{3}}{4} x = x^2 - \frac{4}{3} \\ \Longrightarrow \quad q_3(x) &= \frac{F_3(x)}{\|F_3\|} = \frac{x^2 - 4/3}{\sqrt{\int_{-2}^{2} (x^2 - 4/3)^2 dx}} = \frac{3\sqrt{5}}{16} x^2 - \frac{\sqrt{5}}{4}. \end{aligned}$$

Therefore,

$$q_1(x) = \frac{1}{2}, \; q_2(x) = \frac{\sqrt{3}}{4} x, \text{ and } q_3(x) = \frac{3\sqrt{5}}{16} x^2 - \frac{\sqrt{5}}{4}$$

form an orthonormal basis for the subspace spanned by 1, x, and x^2.

(b) According to (a), we can write

$$x^2 + 2x = \frac{8}{3} q_1(x) + \frac{8}{\sqrt{3}} q_2(x) + \frac{16}{3\sqrt{5}} q_3(x).$$

5. (a) Yes, it is true. Since $\boldsymbol{A}$ is not invertible, we have $|\boldsymbol{A}| = 0$. Then we can have $|\boldsymbol{AB}| = |\boldsymbol{A}|\,|\boldsymbol{B}| = 0 \cdot |\boldsymbol{B}| = 0$. Hence $\boldsymbol{AB}$ is not invertible.

(b) No, it is false. For example, let $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then we have $|\boldsymbol{A} - \boldsymbol{B}| = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1$, and $|\boldsymbol{A}| - |\boldsymbol{B}| = 0 - 0 = 0$. Hence $|\boldsymbol{A} - \boldsymbol{B}| \neq |\boldsymbol{A}| - |\boldsymbol{B}|$.

(c) Yes, it is true. For a skew-symmetric matrix satisfies $\boldsymbol{A}^T = -\boldsymbol{A}$, we have $\left|\boldsymbol{A}^T\right| = |-\boldsymbol{A}|$. Since $\left|\boldsymbol{A}^T\right| = |\boldsymbol{A}|$ and $|-\boldsymbol{A}| = (-1)^n\,|\boldsymbol{A}|$, we can obtain $|\boldsymbol{A}| = (-1)^n\,|\boldsymbol{A}|$. Therefore, if n is odd, we have $|\boldsymbol{A}| = -\,|\boldsymbol{A}|$, which implies $|\boldsymbol{A}| = 0$.

6. (a) Let $S_n = |\boldsymbol{A_n}|$ where $\boldsymbol{A_n}$ is an n by n matrix. For $n \geq 3$, we have

$$S_n = \begin{vmatrix} 3 & 1 & 0 & \cdots & 0 \\ 1 & & & & \\ 0 & & \boldsymbol{A_{n-1}} & & \\ \vdots & & & & \\ 0 & & & & \end{vmatrix} = \begin{vmatrix} 3 & 1 & 0 & 0\cdots & 0 \\ 1 & 3 & 1 & 0\cdots & 0 \\ 0 & 1 & & & \\ 0 & 0 & & \boldsymbol{A_{n-2}} & \\ \vdots & \vdots & & & \\ 0 & 0 & & & \end{vmatrix}.$$

Applying the cofactor formula to the first row, we can have

$$\begin{aligned} S_n &= 3 \cdot (-1)^{1+1}|\boldsymbol{A_{n-1}}| + 1 \cdot (-1)^{1+2} \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & & & & \\ 0 & & \boldsymbol{A_{n-2}} & & \\ \vdots & & & & \\ 0 & & & & \end{vmatrix} \\ &= 3S_{n-1} - 1 \cdot (-1)^{1+1}|\boldsymbol{A_{n-2}}| \quad \text{(apply the cofactor formula to the first column)} \\ &= 3S_{n-1} - S_{n-2}. \end{aligned}$$

Then we can obtain $a = 3$ and $b = -1$.

(b) We have

$$\begin{aligned} S_1 &= 3 \\ S_2 &= 8 \\ S_3 &= 3S_2 - S_1 = 21 \\ S_4 &= 3S_3 - S_2 = 55 \\ S_5 &= 3S_4 - S_3 = 144. \end{aligned}$$

7. (a)

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\begin{aligned}
|\boldsymbol{A}_5| &= \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} \\
&= \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 4 & 4 & 4 & 4 & 4 \end{vmatrix} \quad \text{[add all rows (except the last) to the last row]} \\
&= 4\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} \\
&= 4\begin{vmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} \quad \text{[subtract the last row from each preceding row]} \\
&= 4(-1)(-1)(-1)(-1)(1) \quad \text{[all other terms in the big formula are zero]} \\
&= 4.
\end{aligned}
$$

(b) We have that the $(1,1)$ entry of $\boldsymbol{A}_4^{-1}$ is equal to

$$
(\boldsymbol{A}_4^{-1})_{11} = \frac{\boldsymbol{C}_{11}}{\det(\boldsymbol{A}_4)} = \frac{\det(\boldsymbol{A}_3)}{\det(\boldsymbol{A}_4)} = \frac{\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}} = -\frac{2}{3}.
$$