EE 2030 Linear Algebra Chi-chao Chao
Spring 2012

Final Examination
7:00pm to 10:00pm, June 15, 2012

Problems for Solution:

1. (20%) True or false. (If it is true, prove it. Otherwise, explain why not or find a counterexample.)

   (a) The eigenvalues of $\boldsymbol{A}$ are the same as the eigenvalues of $\boldsymbol{A}^T$.

   (b) The matrix $\boldsymbol{A}$ is similar to the matrix $\boldsymbol{B}$, where

   $$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   (c) The linear transformation $T : \mathcal{R}^3 \to \mathcal{R}^2$ defined by

   $$T(v_1, v_2, v_3) = (v_1 + v_2 + v_3, v_1 + 2v_2 + 3v_3)$$

   has an inverse.

   (d) Given the linear operator T on $\mathcal{R}^3$ defined by

   $$T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} 2v_1 - v_2 \\ -v_1 + 2v_2 - v_3 \\ -v_2 + 2v_3 \end{bmatrix}$$

   there is an orthonormal basis for $\mathcal{R}^3$ such that the matrix representation for T in this basis is a diagonal matrix.

2. (10%) Determine if each of the following matrices is diagonalizable. If it is, find an invertible matrix $\boldsymbol{S}$ and a diagonal matrix $\boldsymbol{\Lambda}$ such that $\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = \boldsymbol{\Lambda}$. If it is not, find its Jordan form.

   (a) $\boldsymbol{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

   (b) $\boldsymbol{A} = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$.

3. (10%) Consider a sequence in which each number is the average of two previous numbers, i.e.,

$$G_{k+2} = \frac{1}{2}\left(G_{k+1} + G_k\right), \text{ for } k \geq 0.$$

Starting from $G_0 = 0$ and $G_1 = 1/2$, find a formula for G_k and compute its limit as $k \to \infty$.

4. (15%) Consider

$$\boldsymbol{A} = \begin{bmatrix} 4 & -1 \\ 5 & 4 \end{bmatrix}.$$

Let

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(a) Is $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$ always positive for every nonzero vector $\boldsymbol{x}$? Why?

(b) Define for every nonzero vector $\boldsymbol{x}$

$$R(\boldsymbol{x}) = \frac{\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}^T\boldsymbol{x}}.$$

Find the maximum of $R(\boldsymbol{x})$, i.e., $\max_{\boldsymbol{x}\neq\boldsymbol{0}} R(\boldsymbol{x})$.

(c) Find a vector $\boldsymbol{x}$ that achieves the minimum of $R(\boldsymbol{x})$ (i.e., $\min_{\boldsymbol{x}\neq\boldsymbol{0}} R(\boldsymbol{x})$).

5. (15%) Prove each of the following statements.

(a) If σ is a (nonzero) singular value of $\boldsymbol{A}$, then there exists a nonzero vector $\boldsymbol{x}$ such that

$$\sigma = \frac{\|\boldsymbol{A}\boldsymbol{x}\|}{\|\boldsymbol{x}\|}.$$

(b) If $\boldsymbol{A}$ is an n by n symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the singular values of $\boldsymbol{A}$ are $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|$.

(c) Let $\boldsymbol{A}$ be an n by n matrix. Then $\boldsymbol{A}^T\boldsymbol{A}$ is similar to $\boldsymbol{A}\boldsymbol{A}^T$. (*Hint:* Consider the singular value decomposition of $\boldsymbol{A}$.)

6. (10%) Consider the vector space M of all 2 by 2 real matrices. The transformation $T : M \to M$ is defined by for every $\boldsymbol{X} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \in M$ by

$$T(\boldsymbol{X}) = \boldsymbol{A}\boldsymbol{X}$$

where

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

(a) Show that T is linear.

(b) In class we know that $\beta = \{\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3, \boldsymbol{V}_4\}$ form a basis for M, where

$$\boldsymbol{V}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ \boldsymbol{V}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ \boldsymbol{V}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ \boldsymbol{V}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Find the matrix representation for T (which is a 4 by 4 matrix) in this basis β.

7. (20%) Consider

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

(a) Find the pseudoinverse of $\boldsymbol{A}$.

(b) Is there a left inverse for $\boldsymbol{A}$? If yes, find one.

(c) Find the projection matrix onto the column space of $\boldsymbol{A}$.

(d) Given

$$\boldsymbol{b} = \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix}$$

there exist $\boldsymbol{p}$ in the column space of $\boldsymbol{A}$ and $\boldsymbol{e}$ in the left nullspace of $\boldsymbol{A}$ such that $\boldsymbol{b} = \boldsymbol{p} + \boldsymbol{e}$. Find the vector $\boldsymbol{x}_r$ in the row space of $\boldsymbol{A}$ such that $\boldsymbol{A}\boldsymbol{x}_r = \boldsymbol{p}$.