EE 2030 Linear Algebra Chi-chao Chao
Spring 2012

Midterm Examination No. 1
7:00pm to 10:00pm, March 30, 2012

Problems for Solution:

1. (25%) Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

   (a) Find the $\boldsymbol{A} = \boldsymbol{LU}$ factorization, where $\boldsymbol{L}$ is a lower triangular matrix with unit diagonal and $\boldsymbol{U}$ is an upper triangular matrix.

   (b) Is $\boldsymbol{A}$ invertible? If yes, find its inverse.

   (c) What is the dimension of the row space of $\boldsymbol{A}$?

   (d) Under what condition on b_1, b_2, b_3, b_4 is the system

$$\boldsymbol{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

   solvable?

   (e) Find all vectors in the left nullspace of $\boldsymbol{A}$.

2. (20%) True or false. (If it is true, prove it. Otherwise, find a counterexample.)

   (a) Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are both n by n skew-symmetric matrices, i.e., $\boldsymbol{A}^T = -\boldsymbol{A}$ and $\boldsymbol{B}^T = -\boldsymbol{B}$. Then $\boldsymbol{ABA}$ is also skew-symmetric.

   (b) Suppose $\boldsymbol{A}$ is an m by n real matrix. If $\boldsymbol{Ax} = \boldsymbol{b}$ always has at least one solution for every $\boldsymbol{b} \in \mathcal{R}^m$, then the only solution to $\boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{0}$ is $\boldsymbol{y} = \boldsymbol{0}$.

   (c) The singular matrices in M form a subspace of M, where M is the vector space of all real 2 by 2 matrices.

   (d) The vectors $(2, 1, -1)$, $(4, 1, 1)$, $(2, -1, 5)$ form a basis for $\mathcal{R}^3$.

3. (20%) Suppose S and T are two subspaces of $\mathcal{R}^4$ given by

$$S = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_4 = 0,\ x_2 + x_3 + x_4 = 0\}$$

   and

$$T = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 + x_4 = 0\}.$$

Consider the intersection $S \cap T$ and the sum $S + T$, where

$$S \cap T = \{\boldsymbol{v} : \boldsymbol{v} \in S \text{ and } \boldsymbol{v} \in T\}$$

and

$$S + T = \{\boldsymbol{s} + \boldsymbol{t} : \boldsymbol{s} \in S,\ \boldsymbol{t} \in T\}.$$

By Problem 1 in Homework Assignment No. 2, we know that both $S \cap T$ and $S + T$ are subspaces of $\mathcal{R}^4$.

(a) Find a basis for S.

(b) Find a basis for T.

(c) What is the dimension of $S \cap T$?

(d) What is the dimension of $S + T$?

4. (15%) Consider the vector space $\mathcal{R}^3$. Decide the linear dependence or independence of

   (a) $(1, 1, 2)$, $(1, 2, 1)$;

   (b) $\boldsymbol{v}_1 - \boldsymbol{v}_2$, $\boldsymbol{v}_2 - \boldsymbol{v}_3$, $\boldsymbol{v}_3 - \boldsymbol{v}_1$ for any $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ in $\mathcal{R}^3$;

   (c) $(1, 1, 0)$, $(1, 0, 0)$, $(0, 1, 1)$, $(2, 3, 4)$.

5. (10%) Under what condition on b_1, b_2, b_3 is this system solvable? Find the complete solution when that condition holds:

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

6. (10%) Write down a matrix with the required property or explain why no such matrix exists.

   (a) The only solution to $\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$.

   (b) A 3 by 4 matrix in the reduced row echelon form which has the vector $\begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$ as a basis for its nullspace.